ON RIGIDITY OF NICHOLS ALGEBRAS

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Abstract. We study deformations of graded braided bialgebras using cohomological methods. In particular, we show that many examples of Nichols algebras, including the finite-dimensional ones arising in the Andruskiewitsch-Schneider program of classification of pointed Hopf algebras, are rigid. This result can be regarded as nonexistence of "braided Lie algebras" with nontrivial bracket.

1. Introduction

Let $\Bbbk$ be a field of characteristic 0 and V a $\Bbbk$-vector space. The symmetric algebra $S(V) = \bigoplus_{n\geq 0} S^n(V)$ is a graded bialgebra by declaring the elements of V *primitive*, i.e. $\Delta(x) = x\otimes 1 + 1\otimes x$ for all $x \in V$, and extending to a morphism of (unital) algebras $\Delta\colon S(V) \to S(V)\otimes S(V)$. Then Lie brackets on V are in one-to-one correspondence with graded deformations of $S(V)$ as a bialgebra (or just as an augmented algebra).

We are interested in graded deformations of bialgebras generalizing $S(V)$, namely, the Nichols algebras of braided vector spaces, which have become prominent in the theory of Hopf algebras (see the survey [1] and references therein). Recall that a *braided vector space* is a vector space V equipped with a linear isomorphism $c\colon V\otimes V \to V\otimes V$ that satisfies the *braid equation*

$$(c\otimes \mathrm{id})(\mathrm{id}\otimes c)(c\otimes \mathrm{id}) = (\mathrm{id}\otimes c)(c\otimes \mathrm{id})(\mathrm{id}\otimes c),$$

where $\mathrm{id} = \mathrm{id}_V$. The *Nichols algebra* of (V, c), denoted by $\mathcal{B}(V, c)$ or just $\mathcal{B}(V)$ if the braiding is clear from the context, is the unique (up to isomorphism) graded braided bialgebra $\mathcal{B} = \bigoplus_{n\geq 0}\mathcal{B}_n$ with $\mathcal{B}_0 = \Bbbk$, $\mathcal{B}_1 = V$ such that the restriction of the braiding of $\mathcal{B}$ to V is c, $\mathcal{B}$ is generated by V as an algebra, and V coincides with the space $P(\mathcal{B})$ of primitive elements of $\mathcal{B}$.

In the case of *symmetric* braiding, i.e., $c^2 = \mathrm{id}$, the concept of braided Lie algebra is well understood [18, 8, 20, 23, 21]. This includes the usual Lie algebras (when c is the flip $v\otimes w \mapsto w\otimes v$), Lie superalgebras (when V is graded by $\mathbb{Z}_2$ and c is the signed flip $v\otimes w \mapsto (-1)^{p(v)p(w)} w\otimes v$ where p denotes parity) and color Lie superalgebras. It follows from Kharchenko's version of PBW Theorem [20, Theorem 7.1] that such Lie structures on (V, c) are in one-to-one correspondence with graded deformations of $\mathcal{B}(V, c)$ as a braided bialgebra with a fixed braiding (see Section 3).

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It is an important and difficult question for what finite-dimensional braided vector spaces the Nichols algebra is also finite-dimensional. This condition puts severe restrictions on c. For example, in the case of signed flip, this happens if and only if the even part of V is zero, in which case the Nichols algebra is the exterior algebra $\Lambda(V)$ and there are no nontrivial graded deformations.

We believe that such rigidity is typical for finite-dimensional Nichols algebras. We establish it for a wide class of symmetric braidings (Theorem 3.3) using the description of finite-dimensional triangular Hopf algebras by Etingof and Gelaki [11, 15, 12]. We also establish a sufficient condition of rigidity (Theorem 5.3) using cohomological techniques, and verify that it is satisfied for finite-dimensional Nichols algebras in the Yetter-Drinfeld category ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ over an abelian group Γ (Theorem 6.3) using a description of these Nichols algebras in terms of generators and relations [4]. It follows that any finite-dimensional Nichols algebra arising from a diagonal braiding, i.e., a braiding of the form $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ where $\{x_1, \dots, x_\theta\}$ is a basis of V and $q_{ij} \in \Bbbk^\times$, does not admit nontrivial graded deformations (Theorem 6.4).

It should be mentioned that the so-called *bosonizations* of these Nichols algebras often admit nontrivial graded deformations (or "liftings"), as has been shown by Andruskiewitsch and Schneider in the course of their program of classification of pointed Hopf algebras [3].

Our sufficient condition also applies to some interesting infinite-dimensional Nichols algebras (see Section 7) and other braided bialgebras close to Nichols algebras (Theorem 7.1). This may explain the difficulty of constructing new examples in [7], where an attempt is made to define and study braided Lie algebras for non-symmetric braiding.

2. Preliminaries

2.1. **Braided tensor categories.** It is often more convenient to work in a category rather than with a stand-alone braided vector space. By a *tensor category* we always mean a strict monoidal $\Bbbk$-linear category, see e.g. [24] for details. We are mostly interested in categories of $\Bbbk$-vector spaces endowed with some additional structure. To simplify notation, we omit associativity isomorphisms and parentheses in tensor products. In particular, we denote the tensor powers of an object V by $V^{\otimes n}$ for all $n \geq 0$, where $V^{\otimes 0}$ is the unit object.

A *braided tensor category* is a tensor category $\mathcal{V}$ with a *braiding*, i.e. a natural family of isomorphisms $c_{V,W} \colon V \otimes W \to W \otimes V$ in $\mathcal{V}$ satisfying the so-called hexagon axioms:

$$c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W) \quad \text{and} \quad c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}),$$

for all U, V, W in $\mathcal{V}$. The braid equation follows:

$$(c_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes \mathrm{id}_W) = (\mathrm{id}_W \otimes c_{U,V})(c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}).$$

The category is said to be *symmetric* if $c_{W,V} c_{V,W} = \mathrm{id}_{V\otimes W}$ for all V, W in $\mathcal{V}$.

The most well known braided tensor categories are the category of (co)modules over a (co)quasitriangular bialgebra and the category of Yetter-Drinfeld modules over a Hopf algebra with bijective antipode. We will now briefly recall the relevant definitions and fix notation; details can be found in textbooks such as [27, 22]. We use the standard Sweedler notation for coalgebras and comodules.

A *coquasitriangular (CQT) bialgebra* is a pair (H,β) where H is a bialgebra and β is a bilinear form $H\times H\to \Bbbk$ that is invertible with respect to convolution and satisfies

$$\begin{aligned}\beta(h_{(1)},k_{(1)})h_{(2)}k_{(2)} &= \beta(h_{(2)},k_{(2)})k_{(1)}h_{(1)},\\ \beta(hk,\ell) &= \beta(h,\ell_{(1)})\beta(k,\ell_{(2)}),\\ \beta(\ell,hk) &= \beta(\ell_{(2)},h)\beta(\ell_{(1)},k),\end{aligned}$$

for all $h,k,\ell\in H$. The category of right comodules $\mathcal{M}^H$ is braided as follows:

$$c_{V,W}(v\otimes w)=\beta(v_{(1)},w_{(1)})w_{(0)}\otimes v_{(0)}, \qquad \text{for all } v\in V, w\in W. \tag{1}$$

Similarly, the category of left comodules ${}^H\mathcal{M}$ is braided by

$$c_{V,W}(v\otimes w)=\beta(w_{(-1)},v_{(-1)})w_{(0)}\otimes v_{(0)}, \qquad \text{for all } v\in V, w\in W.$$

If G is a group then the Hopf algebra $H=\Bbbk G$ admits a CQT structure β if and only if G is abelian. In this case the possible maps β are just linear extensions of bicharacters $G\times G\to\Bbbk^\times$. Right H-comodules are just G-graded vector spaces, $V=\bigoplus_{g\in G}V_g$, and the braiding is given by $v\otimes w\mapsto \beta(g,h)w\otimes v$ for all $v\in V_g$, $w\in W_h$, $g,h\in G$.

An object V of the Yetter-Drinfeld category ${}^H_H\mathcal{YD}$ is simultaneously a left module and a left comodule such that the following compatibility condition holds:

$$h_{(1)}v_{(-1)}\otimes h_{(2)}\cdot v_{(0)} = (h_{(1)}\cdot v)_{(-1)}h_{(2)}\otimes (h_{(1)}\cdot v)_{(0)} \qquad \text{for all } v\in V,\ h\in H.$$

A morphism is a linear map preserving both action and coaction. The braiding is given by

$$c_{V,W}\colon v\otimes w\mapsto v_{(-1)}\cdot w\otimes v_{(0)}.$$

The category of right Yetter-Drinfeld modules $\mathcal{YD}^H_H$ is defined in a similar manner. If Γ is a group and $H=\Bbbk\Gamma$ then an object in ${}^H_H\mathcal{YD}$ is just a Γ-graded vector space with a left action of Γ such that $g\cdot V_h=V_{ghg^{-1}}$, for all $g,h\in\Gamma$. The braiding is given by $v\otimes w\mapsto g\cdot w\otimes v$, for all $v\in V_g$, $w\in W$. In particular, if Γ is abelian then the semisimple objects in $\mathcal{YD}^H_H$ are vector spaces graded by the direct product $\Gamma\times\widehat{\Gamma}$ where $\widehat{\Gamma}$ is the character group of Γ. For a vector space V with such a grading, we will denote the homogeneous component of degree (g,χ) by V_g^χ. The braiding becomes $v\otimes w\mapsto\psi(g)w\otimes v$, for all $v\in V_g^\chi$ and $w\in W_h^\psi$.

If a CQT bialgebra (H,β) is a Hopf algebra then its antipode is bijective. Moreover $\mathcal{M}^H$ can be regarded as a full subcategory of the Yetter-Drinfeld category $\mathcal{YD}^H_H$ if we define the right action of H on a right comodule V by means of the usual left action of H^* and the homomorphism of algebras $H^{\mathrm{op}}\to H^*\colon h\mapsto\beta(\cdot,h)$, i.e., $v\cdot h=\sum\beta(v_{(1)},h)v_{(0)}$, for all $v\in V$, $h\in H$. Similarly, ${}^H\mathcal{M}$ can be regarded as a full subcategory of ${}^H_H\mathcal{YD}$.

If (U,c) is a finite-dimensional braided vector space then the FRT construction [22, 29] yields a CQT bialgebra (H,β) such that $U\in\mathcal{M}^H$ and $c=c_{U,U}$ where $c_{U,U}$ is given by (1). Moreover, for any $V,W\in\mathcal{M}^H$ and a linear map $f\colon V\to W$ that *commutes with the braiding with* U in the sense that $(f\otimes\mathrm{id})c_{U,V}=c_{U,W}(\mathrm{id}\otimes f)$ and $(\mathrm{id}\otimes f)c_{V,U}=c_{W,U}(f\otimes\mathrm{id})$, there exists a biideal I of H contained in the left and right kernels of the bilinear form β such that f is a morphism in $\mathcal{M}^{H/I}$ [29, Corollary 1.9]. Hence, replacing (H,β) by $(\bar H,\bar\beta)$, where $\bar H$ is the quotient of H by the largest biideal contained in the left and right kernels of β and where $\bar\beta$ is induced by β, we obtain a braided category, $\mathcal{M}^{\bar H}$, that contains (U,c) and all linear maps that commute with the braiding with U.

There is a Hopf algebra version of the above construction — see e.g. [29] and references therein — for braided vector spaces satisfying a certain condition, called *rigidity* in [29], which allows us to define the braiding operators c_{U,U^*}, $c_{U^*,U}$ and c_{U^*,U^*}, where U^* is the dual space. Namely, there exists a CQT Hopf algebra (H,β) such that $U \in \mathcal{M}^H$ and $c = c_{U,U}$. Again, any linear map that commutes with the braiding with U can be included in the category $\mathcal{M}^{H/I}$ where I is a Hopf ideal contained in the left and right kernels of β, see the proof of [29, Proposition 5.4]. Since the largest biideal contained in the kernels of β is automatically a Hopf ideal, we obtain a CQT Hopf algebra $\bar{H}$ such that $\mathcal{M}^{\bar{H}}$ includes (U,c) and all linear maps that commute with the braiding with U.

We are especially interested in the case of diagonal braiding: $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$ where $\{x_1,\dots,x_\theta\}$ is a basis of U and $q_{ij} \in \Bbbk^\times$. Here we can take $H = \Bbbk G$, where G is the free abelian group $\mathbb{Z}^\theta$, and define the bicharacter β by setting $\beta(e_i,e_j) = q_{ij}$, where $\{e_1,\dots,e_\theta\}$ is the standard basis of $\mathbb{Z}^\theta$. If we make U a G-graded vector space by declaring $x_i \in U_{e_i}$ then we get $c = c_{U,U}$ in $\mathcal{M}^H$. Alternatively, we can make U an object of ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ for each abelian group Γ containing elements $g_1,\dots,g_\theta$ such that there exist characters $\chi_1,\dots,\chi_\theta \in \widehat{\Gamma}$ satisfying $\chi_j(g_i) = q_{ij}$; then we declare $x_i \in U_{g_i}^{\chi_i}$ and get $c = c_{U,U}$ in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$. We can choose the group Γ so that it is generated by $g_1,\dots,g_\theta$ and the characters $\chi_1,\dots,\chi_\theta$ separate points of Γ. It is easy to see that in this case a linear map $f\colon V \to W$ commutes with the braiding with U if and only if f is a morphism in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$.

2.2. **Braided bialgebras.** A *bialgebra* in a braided tensor category $\mathcal{V}$ with unit object $\mathbb{1}$ is an object $\mathcal{B}$ with four morphisms: multiplication $m\colon \mathcal{B}\otimes\mathcal{B} \to \mathcal{B}$, unit $u\colon \mathbb{1} \to \mathcal{B}$, comultiplication $\Delta\colon \mathcal{B} \to \mathcal{B}\otimes\mathcal{B}$ and counit $\varepsilon\colon \mathcal{B} \to \mathbb{1}$ such that $(\mathcal{B},m,u)$ is a unital algebra, $(\mathcal{B},\Delta,\varepsilon)$ is a counital coalgebra, and the following compatibility conditions hold:

$$\Delta m = (m\otimes m)(\mathrm{id}_{\mathcal{B}} \otimes c_{\mathcal{B},\mathcal{B}} \otimes \mathrm{id}_{\mathcal{B}})(\Delta\otimes\Delta), \quad \varepsilon u = \mathrm{id}_{\mathbb{1}}, \quad \varepsilon m = \varepsilon\otimes\varepsilon, \quad \Delta u = u\otimes u.$$

Note that the braiding appears only in the compatibility condition involving m and Δ.

One can define a *braided bialgebra* without reference to any categories [29]: it is a braided vector space $(\mathcal{B},c)$ with four linear maps, $m\colon \mathcal{B}\otimes\mathcal{B} \to \mathcal{B}$, $u\colon \Bbbk \to \mathcal{B}$, $\Delta\colon \mathcal{B} \to \mathcal{B}\otimes\mathcal{B}$ and $\varepsilon\colon \mathcal{B} \to \Bbbk$, that commute with the braiding induced by c among the tensor powers of $\mathcal{B}$ and satisfy the following conditions: $(\mathcal{B},m,u)$ is a unital algebra, $(\mathcal{B},\Delta,\varepsilon)$ is a counital coalgebra, u is a counital coalgebra map, ε is a unital algebra map, and finally $\Delta m = (m\otimes m)(\mathrm{id}_{\mathcal{B}} \otimes c \otimes \mathrm{id}_{\mathcal{B}})(\Delta\otimes\Delta)$.

Obviously, a bialgebra $\mathcal{B}$ in a braided tensor category consisting of vector spaces and linear maps (such as $\mathcal{M}^H$ or ${}^H_H\mathcal{YD}$) satisfies the definition of braided bialgebra with $c = c_{\mathcal{B},\mathcal{B}}$. Conversely, it is shown in [29] that any finite-dimensional braided bialgebra $(\mathcal{B},m,u,\Delta,\varepsilon,c)$ can be included in the category $\mathcal{M}^H$ over a suitable CQT bialgebra (Hopf algebra if c is rigid) H such that m,u,Δ,ε are morphisms in $\mathcal{M}^H$ and $c = c_{\mathcal{B},\mathcal{B}}$ in $\mathcal{M}^H$.

We are mainly interested in the case of the Nichols algebra $\mathcal{B}(V)$ of a finite-dimensional vector space V with a rigid braiding c, which is a braided Hopf algebra, not necessarily finite-dimensional but equipped with a grading over non-negative integers whose components are finite-dimensional. It can be constructed as the quotient of the tensor algebra $T(V)$ by a graded biideal $\mathcal{I}(V)$ [1, Proposition 2.2], which is determined by the braiding c; indeed the homogeneous components of $\mathcal{I}(V)$ are the kernels of the so-called *quantum symmetrizers* on the tensor powers of V [1, Proposition 2.11]. This construction can be carried out either with the stand-alone braided vector space (V,c) or in a suitable braided category of comodules or Yetter-Drinfeld modules.

2.3. **Graded deformations and liftings.** We review the theory of formal graded deformations and liftings from [25], but in a slightly more general setting. The theory of formal bialgebra deformations was introduced by Gerstenhaber and Schack [16], while the graded version and its connection to liftings was considered by Du, Chen and Ye [10]. In this context, a *graded bialgebra* will mean a bialgebra $\mathcal{B}$ in a braided tensor category $\mathcal{V}$ (consisting of vector spaces and linear maps) equipped with a grading, as an object in $\mathcal{V}$, over non-negative integers, $\mathcal{B} = \bigoplus_{n\geq 0} \mathcal{B}_n$, which is at the same time an algebra and a coalgebra grading, i.e., $\mathcal{B}_i\mathcal{B}_j \subseteq \mathcal{B}_{i+j}$ and $\Delta(\mathcal{B}_k) \subseteq \bigoplus_{i+j=k} \mathcal{B}_i \otimes \mathcal{B}_j$, for all $i,j,k \geq 0$.

Let t be an indeterminate and consider the polynomial algebra $\Bbbk[t]$ equipped with its standard grading, i.e., t has degree 1. By extending scalars from $\Bbbk$ to $\Bbbk[t]$, the braided tensor category $\mathcal{V}$ gives rise to the braided tensor category $\mathcal{V}_{\Bbbk[t]}$. A (formal) *graded deformation* of a graded bialgebra $(\mathcal{B}, m, \Delta)$ in $\mathcal{V}$ is a $\Bbbk[t]$-linear graded structure (m_t, Δ_t) on $\mathcal{B}[t] = \mathcal{B}\otimes\Bbbk[t]$ such that $(\mathcal{B}[t], m_t, \Delta_t)$ is a graded bialgebra in $\mathcal{V}_{\Bbbk[t]}$.

We say that two graded deformations, $(\mathcal{B}[t], m_t, \Delta_t)$ and $(\mathcal{B}[t], m'_t, \Delta'_t)$, are *equivalent* if there exists a $\Bbbk[t]$-linear graded bialgebra isomorphism $f\colon (\mathcal{B}[t], m_t, \Delta_t) \to (\mathcal{B}[t], m'_t, \Delta'_t)$.

A *lifting* $(\mathcal{U}, \pi)$ of $\mathcal{B}$ consists of a filtered bialgebra $\mathcal{U}$ and a filtered vector space isomorphism $\pi\colon \mathcal{U} \to \mathcal{B}$ such that $\operatorname{gr}\pi\colon \operatorname{gr}\mathcal{U} \to \operatorname{gr}\mathcal{B} = \mathcal{B}$ is an isomorphism of graded bialgebras. An *equivalence* between liftings $(\mathcal{U}, \pi)$ and $(\mathcal{U}', \pi')$ is a filtered bialgebra isomorphism $f\colon \mathcal{U} \to \mathcal{U}'$ such that $\operatorname{gr}\pi \circ \operatorname{gr} f = \operatorname{gr}\pi'$.

A graded deformation is given by a sequence of pairs of maps (m_i, Δ_i), $i \geq 0$, of degree $-i$ such that $m_t|_{\mathcal{B}\otimes\mathcal{B}} = m + \sum_{i\geq 1} m_i t^i$ and $\Delta_t|_{\mathcal{B}} = \Delta + \sum_{i\geq 1} \Delta_i t^i$. We also denote $(m_0, \Delta_0) = (m, \Delta)$. A graded deformation $(\mathcal{B}[t], m_t, \Delta_t)$ defines a lifting $(\mathcal{U}, \pi)$, where $\mathcal{U}$ is $\mathcal{B}$ as a filtered vector space, π is identity, and $(m_{\mathcal{U}}, \Delta_{\mathcal{U}}) = (m_t, \Delta_t)|_{t=1}$.

If $(\mathcal{U}, \pi)$ is a lifting, then the linear maps $\tilde{m}\colon \mathcal{B}\otimes\mathcal{B} \overset{\pi^{-1}\otimes\pi^{-1}}{\longrightarrow} \mathcal{U}\otimes\mathcal{U} \overset{m_{\mathcal{U}}}{\to} \mathcal{U} \overset{\pi}{\to} \mathcal{B}$ and $\tilde{\Delta}\colon \mathcal{B} \overset{\pi^{-1}}{\to} \mathcal{U} \overset{\Delta_{\mathcal{U}}}{\to} \mathcal{U}\otimes\mathcal{U} \overset{\pi\otimes\pi}{\longrightarrow} \mathcal{B}\otimes\mathcal{B}$ decompose into direct sums of homogeneous components m_i, Δ_i of degrees $-i$ for $i \geq 0$, and the structure maps $(m_t, \Delta_t) = (\sum_i m_i t^i, \sum_i \Delta_i t^i)$ on $\mathcal{B}[t]$ define a formal graded deformation of $\mathcal{B}$.

Up to equivalence, these correspondences are inverses of each other.

2.4. **Graded bialgebra cohomology.** Let $\mathcal{B}$ be a bialgebra in $\mathcal{V}$. Consider the bisimplicial complex $\mathbf{B} = (\mathbf{B}^{p,q})_{p,q\geq 0}$,

$$\mathbf{B}^{p,q} = \operatorname{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^{\otimes q}).$$

The left and right diagonal actions and coactions of $\mathcal{B}$ on $\mathcal{B}^{\otimes n}$ will be denoted by $\lambda_l, \lambda_r, \rho_l, \rho_r$, respectively. Note that they involve the braiding. The horizontal faces

$$\partial_i^h\colon \operatorname{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^{\otimes q}) \to \operatorname{Hom}(\mathcal{B}^{\otimes (p+1)}, \mathcal{B}^{\otimes q})$$

and degeneracies

$$\sigma_i^h\colon \operatorname{Hom}(\mathcal{B}^{\otimes (p+1)}, \mathcal{B}^{\otimes q}) \to \operatorname{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^{\otimes q})$$

are those for computing Hochschild cohomology:

$$\begin{aligned}\partial_0^h f &= \lambda_l(\mathrm{id}\otimes f),\\ \partial_i^h f &= f(\mathrm{id}\otimes\ldots\otimes m\otimes\ldots\otimes\mathrm{id}),\ 1\le i\le p,\\ \partial_{p+1}^h f &= \lambda_r(f\otimes\mathrm{id}),\\ \sigma_i^h f &= f(\mathrm{id}\otimes\ldots\otimes u\otimes\ldots\otimes\mathrm{id});\end{aligned}$$

the vertical faces

$$\partial_j^c\colon \mathrm{Hom}(\mathcal{B}^{\otimes p},\mathcal{B}^{\otimes q})\to\mathrm{Hom}(\mathcal{B}^{\otimes p},\mathcal{B}^{\otimes (q+1)})$$

and degeneracies

$$\sigma_j^c\colon \mathrm{Hom}(\mathcal{B}^{\otimes p},\mathcal{B}^{\otimes (q+1)})\to\mathrm{Hom}(\mathcal{B}^{\otimes p},\mathcal{B}^{q})$$

are those for computing coalgebra (Cartier) cohomology:

$$\begin{aligned}\partial_0^c f &= (\mathrm{id}\otimes f)\rho_l,\\ \partial_j^c f &= (\mathrm{id}\otimes\ldots\otimes\Delta\otimes\ldots\otimes\mathrm{id})f,\quad 1\le j\le q,\\ \partial_{q+1}^c f &= (f\otimes\mathrm{id})\rho_r,\\ \sigma_i^c f &= (\mathrm{id}\otimes\ldots\otimes\varepsilon\otimes\ldots\otimes\mathrm{id})f.\end{aligned}$$

The vertical and horizontal differentials are given by the usual alternating sums

$$\partial^h=\sum(-1)^i\partial_i^h,\qquad \partial^c=\sum(-1)^j\partial_j^c.$$

By abuse of notation we identify a cosimplicial bicomplex with its associated cochain bicomplex. The *bialgebra cohomology* of $\mathcal{B}$ is then defined as

$$\mathrm{H}_{\mathrm{b}}^*(\mathcal{B})=\mathrm{H}^*(\mathrm{Tot}\,\mathbf{B}).$$

where

$$\mathrm{Tot}\,\mathbf{B}=\mathbf{B}^{0,0}\to\mathbf{B}^{1,0}\oplus\mathbf{B}^{0,1}\to\ldots\to\bigoplus_{p+q=n}\mathbf{B}^{p,q}\overset{\partial^b}{\to}\ldots$$

and ∂^b is given by the sign trick (i.e., $\partial^b|_{\mathbf{B}^{p,q}}=\partial^h\oplus(-1)^p\partial^c\colon\mathbf{B}^{p,q}\to\mathbf{B}^{p+1,q}\oplus\mathbf{B}^{p,q+1}$).

Let $\mathbf{B}_0$ denote the bicomplex obtained from $\mathbf{B}$ by replacing the edges by zeroes, i.e., $\mathbf{B}_0^{p,0}=0=\mathbf{B}_0^{0,q}$ for all p,q. The *truncated bialgebra cohomology* is

$$\widehat{\mathrm{H}}_{\mathrm{b}}^*(\mathcal{B})=\mathrm{H}^{*+1}(\mathrm{Tot}\,\mathbf{B}_0).$$

For computations, it is convenient to use the normalized bicomplex $\mathbf{B}^+$, which is obtained from the cochain bicomplex $\mathbf{B}$ by replacing $\mathbf{B}^{p,q}=\mathrm{Hom}(\mathcal{B}^{\otimes p},\mathcal{B}^{\otimes q})$ with the intersection of degeneracies

$$(\mathbf{B}^+)^{p,q}=(\cap\,\mathrm{Ker}\,\sigma_i^h)\cap(\cap\,\mathrm{Ker}\,\sigma_j^c)\simeq\mathrm{Hom}((\mathcal{B}^+)^{\otimes p},(\mathcal{B}^+)^{\otimes q}),$$

where $\mathcal{B}^+=\ker(\varepsilon)$. This change does not affect the cohomology.

We can describe the first two cohomology groups as follows:

$$\widehat{\mathrm{H}}_{\mathrm{b}}^1(\mathcal{B})=\{f\colon\mathcal{B}^+\to\mathcal{B}^+\mid f(ab)=af(b)+f(a)b,\ \Delta f(a)=a_{(1)}\otimes f(a_{(2)})+f(a_{(1)})\otimes a_{(2)}\}$$

and

$$\widehat{\mathrm{H}}_{\mathrm{b}}^2(\mathcal{B})=\widehat{\mathrm{Z}}_{\mathrm{b}}^2(\mathcal{B})/\widehat{\mathrm{B}}_{\mathrm{b}}^2(\mathcal{B}),$$

where

$$\widehat{\mathrm{Z}}^2_{\mathrm{b}}(\mathcal{B}) = \big\{(f,g) \bigm| f\colon \mathcal{B}^+\otimes\mathcal{B}^+ \to \mathcal{B}^+,\ g\colon \mathcal{B}^+ \to \mathcal{B}^+\otimes\mathcal{B}^+,$$

$$af(b,c) + f(a,bc) = f(ab,c) + f(a,b)c, \tag{2}$$

$$c_{(1)}\otimes g(c_{(2)}) + (\mathrm{id}\otimes\Delta)g(c) = (\Delta\otimes\mathrm{id})g(c) + g(c_{(1)})\otimes c_{(2)}, \tag{3}$$

$$(f\otimes m)\Delta(a\otimes b) - \Delta f(a,b) + (m\otimes f)\Delta(a\otimes b) = - (\Delta a)g(b) + g(ab) - g(a)(\Delta b)\big\} \tag{4}$$

and

$$\widehat{\mathrm{B}}^2_{\mathrm{b}}(\mathcal{B}) = \big\{(f,g) \bigm| \exists h\colon \mathcal{B}^+ \to \mathcal{B}^+,\ f(a,b) = ah(b) - h(ab) + h(a)b,$$
$$g(c) = -c_{(1)}\otimes h(c_{(2)}) + \Delta h(c) - h(c_{(1)})\otimes c_{(2)}\big\},$$

where the elements a, b, c range over $\mathcal{B}^+$. All maps above are assumed to be morphisms in $\mathcal{V}$. By $\Delta(a\otimes b)$ we mean the braided coproduct in $\mathcal{B}\otimes\mathcal{B}$, namely, $(\mathrm{id}\otimes c_{\mathcal{B},\mathcal{B}}\otimes\mathrm{id})(a_{(1)}\otimes a_{(2)}\otimes b_{(1)}\otimes b_{(2)})$, and we write $f(-,-)$ instead of $f(-\otimes-)$. In the resulting deformation (see the next subsection), Equation (2) will correspond to associativity, Equation (3) to coassociativity and Equation (4) to compatibility.

Now assume that $\mathcal{B}$ is $\mathbb{Z}$-graded and let $\mathbf{B}_\ell$ denote the subcomplex of $\mathbf{B}$ consisting of homogeneous maps of degree ℓ, i.e.,

$$\mathbf{B}^{p,q}_\ell = \mathrm{Hom}(\mathcal{B}^{\otimes p}, \mathcal{B}^{\otimes q})_\ell = \{f\colon \mathcal{B}^{\otimes p} \to \mathcal{B}^{\otimes q} \mid f \text{ is homogeneous of degree } \ell\}.$$

Complexes $(\mathbf{B}_0)_\ell$, $\mathbf{B}^+_\ell$ and $(\mathbf{B}^+_0)_\ell$ are defined analogously. The graded bialgebra and truncated graded bialgebra cohomologies are then defined by:

$$\mathrm{H}^*_{\mathrm{b}}(\mathcal{B})_\ell = \mathrm{H}^*(\mathrm{Tot}\,\mathbf{B}_\ell) = \mathrm{H}^*(\mathrm{Tot}\,\mathbf{B}^+_\ell),$$
$$\widehat{\mathrm{H}}^*_{\mathrm{b}}(\mathcal{B})_\ell = \mathrm{H}^{*+1}(\mathrm{Tot}(\mathbf{B}_0)_\ell) = \mathrm{H}^{*+1}(\mathrm{Tot}(\mathbf{B}^+_0)_\ell).$$

Note that if the support of the grading is finite, in particular if $\mathcal{B}$ is finite-dimensional, then

$$\mathrm{H}^*_{\mathrm{b}}(\mathcal{B}) = \bigoplus_{\ell\in\mathbb{Z}} \mathrm{H}^*_{\mathrm{b}}(\mathcal{B})_\ell \quad\text{and}\quad \widehat{\mathrm{H}}^*_{\mathrm{b}}(\mathcal{B}) = \bigoplus_{\ell\in\mathbb{Z}} \widehat{\mathrm{H}}^*_{\mathrm{b}}(\mathcal{B})_\ell.$$

2.5. **Cohomological aspects of graded deformations.** Given a graded deformation of $\mathcal{B}$, let r be the smallest positive integer for which $(m_r, \Delta_r) \neq (0,0)$ (if such an r exists). Then (m_r, Δ_r) is a 2-cocycle in $\widehat{\mathrm{Z}}^2_{\mathrm{b}}(\mathcal{B})_{-r}$. Every nontrivial deformation is equivalent to one for which the corresponding (m_r, Δ_r) represents a nontrivial cohomology class [16, 10]. Hence, if $\widehat{\mathrm{H}}^2_{\mathrm{b}}(\mathcal{B})_{(\ell)} = 0$ for all $\ell < 0$, then $\mathcal{B}$ is *rigid*, i.e., has no nontrivial graded deformations.

Conversely, given a positive integer r and a 2-cocycle (m', Δ') in $\widehat{\mathrm{Z}}^2_{\mathrm{b}}(\mathcal{B})_{-r}$, the maps $m + t^r m'$ and $\Delta + t^r\Delta'$ define a bialgebra structure on $\mathcal{B}[t]/(t^{r+1})$ over $\Bbbk[t]/(t^{r+1})$. There may or may not exist (m_{r+k}, Δ_{r+k}), $k \geq 1$, for which $m_t = m + t^r m' + \sum_{k\geq 1} t^{r+k} m_{r+k}$ and $\Delta_t = \Delta + t^r\Delta' + \sum_{k\geq1} t^{r+k}\Delta_{r+k}$ make $\mathcal{B}[t]$ into a bialgebra over $\Bbbk[t]$.

An *r-deformation* of $\mathcal{B}$ is a graded deformation of $\mathcal{B}$ over $\Bbbk[t]/(t^{r+1})$, i.e. a pair (m^r_t, Δ^r_t) defining a bialgebra structure on $\mathcal{B}[t]/(t^{r+1})$ over $\Bbbk[t]/(t^{r+1})$ such that $(m^r_t, \Delta^r_t)|_{t=0} = (m, \Delta)$. For any 2-cocycle (m', Δ') in $\widehat{\mathrm{Z}}^2_{\mathrm{b}}(\mathcal{B})_{-r}$, there exists an r-deformation, given by $(m + t^r m', \Delta + t^r\Delta')$.

If a given $(r-1)$-deformation can be extended to an r-deformation, then all ways of doing so are parametrized by $\widehat{\mathrm{H}}^2_{\mathrm{b}}(\mathcal{B})_{-r}$. More precisely, suppose that $(\mathcal{B}[t]/(t^r),\ m_t^{r-1},\ \Delta_t^{r-1})$ is an $(r-1)$-deformation, where

$$m_t^{r-1} = m + tm_1 + \ldots + t^{r-1}m_{r-1}, \qquad \Delta_t^{r-1} = \Delta + t\Delta_1 + \ldots + t^{r-1}\Delta_{r-1}.$$

If

$$D = (\mathcal{B}[t]/(t^{r+1}), m_t^{r-1} + t^r m_r, \Delta_t^{r-1} + t^r \Delta_r)$$

is an r-deformation, then

$$D' = (\mathcal{B}[t]/(t^{r+1}), m_t^{r-1} + t^r m'_r, \Delta_t^{r-1} + t^r \Delta'_r)$$

is an r-deformation if and only if $(m'_r - m_r, \Delta'_r - \Delta_r) \in \widehat{\mathrm{Z}}^2_{\mathrm{b}}(\mathcal{B})_{-r}$. Note also that if $(m'_r - m_r, \Delta'_r - \Delta_r) \in \widehat{\mathrm{B}}^2_{\mathrm{b}}(\mathcal{B})_{-r}$, then deformations D and D' are equivalent.

The obstruction to extend r-deformations to $(r+1)$-deformations lies in $\widehat{\mathrm{H}}^3_{\mathrm{b}}(\mathcal{B})_{-r}$.

3. The case of symmetric braiding

Let (V, c) be a braided vector space with $c^2 = \mathrm{id}$. Then $\mathcal{B}(V)$ is a quadratic algebra: it is the quotient of $T(V)$ by the ideal generated by the elements $x \otimes y - c(x \otimes y)$, for $x, y \in V$. If c is the flip (respectively, signed flip) then $\mathcal{B}(V) = S(V)$ (respectively, $S(V_0) \otimes \Lambda(V_1)$) and the graded deformations of $\mathcal{B}(V)$ are in one-to-one correspondence with brackets $[\,,\,]\colon V \otimes V \to V$ making V a Lie algebra (respectively, superalgebra). For arbitrary c, we need the following generalization of Lie algebra introduced by Gurevich [18] under the name "Lie c-algebra".

Definition 3.1. *Let L be a vector space, $c\colon L \otimes L \to L \otimes L$ a symmetric braiding, and $[\,,\,]\colon L \otimes L \to L$ a linear map. Then $(L, [\,,\,], c)$ is a* braided Lie algebra *if*

$$c([\,,\,] \otimes \mathrm{id}_L) = (\mathrm{id}_L \otimes [\,,\,])(c \otimes \mathrm{id}_L)(\mathrm{id}_L \otimes c) \qquad \textit{(compatibility)},$$

$$[\,,\,](\mathrm{id}_{L\otimes L} + c) = 0 \qquad \textit{(anticommutativity)}$$

$$[\,,\,]([\,,\,] \otimes \mathrm{id}_L)\Big(\mathrm{id}_{L\otimes L\otimes L} + (c \otimes \mathrm{id}_L)(\mathrm{id}_L \otimes c) + (c \otimes \mathrm{id}_L)(\mathrm{id}_L \otimes c)\Big) = 0 \qquad \textit{(Jacobi identity).}$$

Note that the compatibility condition (together with $c^2 = \mathrm{id}$) simply means that the bracket commutes with c, and the above Jacobi identity implies a similar identity for $[\,,\,](\mathrm{id}_L \otimes [\,,\,])$ instead of $[\,,\,]([\,,\,] \otimes \mathrm{id}_L)$. It is straightforward to check that if a vector space A is equipped with a symmetric braiding c and an associative product $m\colon A \otimes A \to A$ that commutes with c then $(A, [\,,\,]_c, c)$ is a braided Lie algebra, where $[\,,\,]_c$ is the *braided commutator* $m(\mathrm{id}_{A\otimes A} - c)$.

Braided Lie algebras naturally arise as Lie algebras in a symmetric tensor category $\mathcal{V}$. A Lie algebra in $\mathcal{V}$ is an object L endowed with a morphism $[\,,\,]\colon L \otimes L \to L$ such that the anticommutativity and Jacobi identity hold for $c = c_{L,L}$. If (H, β) is a *cotriangular* bialgebra (i.e., a CQT bialgebra satisfying $\beta^{-1}(h, k) = \beta(k, h)$ for all $h, k \in H$) then the category $\mathcal{M}^H$ is symmetric; Lie algebras in this category were introduced and studied in [8, 9] under the name (H, β)-*Lie algebras*. By an argument similar to [29] (see Subsection 2.2 above), any finite-dimensional braided Lie algebra can be regarded as an (H, β)-Lie algebra for a suitable cotriangular bialgebra (Hopf algebra if the braiding is rigid).

Given a braided Lie algebra $(L, [\ ,\ ], c)$, the *universal enveloping algebra*, which we will denote $\mathcal{U}_c(L)$, is the quotient of the tensor algebra $T(L)$ by the ideal generated by the degree 2 elements $x \otimes y - c(x \otimes y) - [x, y]$ where $x, y \in L$. The usual increasing filtration of $T(L)$ gives rise to the *standard filtration* of $\mathcal{U}_c(L)$. As one would expect, $\mathcal{U}_c(L)$ becomes a braided bialgebra if we declare the elements of L primitive. It is not true in general that, given an ordered basis of L, the corresponding PBW monomials form a basis of $\mathcal{U}_c(L)$. However, the following version of PBW Theorem holds.

Theorem 3.2. [20, Theorem 7.1] *The graded algebra* $\operatorname{gr}\mathcal{U}_c(L)$ *associated to the standard filtration of* $\mathcal{U}_c(L)$ *is naturally isomorphic to* $\mathcal{U}_c(L^\circ)$ *where* L° *denotes the braided Lie algebra with the same underlying braided vector space as* L *but with zero bracket.* □

The standard filtration of $\mathcal{U}_c(L)$ coincides with its coradical filtration. Also $\mathcal{U}_c(L^\circ) = \mathcal{B}(L, c)$.

It follows that graded deformations of $\mathcal{B}(V, c)$ as a braided augmented algebra or as a braided bialgebra (with a fixed braiding) are in one-to-one correspondence with brackets on V making it a braided Lie algebra. Here the "graded deformations" and "braided Lie algebras" can be understood in the sense of a stand-alone object or an object in $\mathcal{M}^H$ for a suitable cotriangular bialgebra (H, β).

For $H = \Bbbk G$, where G is an abelian group, the cotriangular structures on H are linear extensions of skew-symmetric bicharacters $\beta\colon G \times G \to \Bbbk^\times$. In this case the (H, β)-Lie algebras are known as the *color Lie superalgebras with grading group* G *and commutation factor* β. Note that the braiding is diagonal and, conversely, any braided Lie algebra with a diagonal braiding can be regarded as a color Lie superalgebra for some G and β.

By a trick going back to Scheunert [28], color Lie superalgebras can be twisted to become ordinary Lie superalgebras. This procedure works in the same way for all color Lie superalgebras with given G and β, and is associated to a suitable cocycle twist of $(\Bbbk G, \beta)$ as a CQT bialgebra. Recall that a *right 2-cocycle* on a bialgebra H is a convolution-invertible map $\sigma\colon H \otimes H \to \Bbbk$ satisfying the following equations for all $h, k, \ell \in H$:

$$\sigma(h, k_{(1)}\ell_{(1)})\sigma(k_{(2)}, \ell_{(2)}) = \sigma(h_{(1)}k_{(1)}, \ell)\sigma(h_{(2)}, k_{(2)}), \qquad \sigma(h, 1) = \sigma(1, h) = \varepsilon(h).$$

Also recall that if (H, β) is a cotriangular (more generally, CQT) bialgebra then (H_σ, β_σ) is again a cotriangular (respectively, CQT) bialgebra, see e.g. [22]; here $H_\sigma = H$ as a coalgebra, the multiplication of H_σ is given by

$$h \cdot_\sigma k = \sigma^{-1}(h_{(1)}, k_{(1)})h_{(2)}k_{(2)}\sigma(h_{(3)}, k_{(3)}),$$

and

$$\beta_\sigma(h, k) = \sigma^{-1}(k_{(1)}, h_{(1)})\beta(h_{(2)}k_{(2)})\sigma(h_{(3)}, k_{(3)}).$$

Moreover, σ yields an equivalence of braided tensor categories $\mathcal{M}^H$ and $\mathcal{M}^{H_\sigma}$, which is the identity on objects and morphisms and only transforms the tensor product. If A is an algebra (not necessarily associative) in $\mathcal{M}^H$ with multiplication $m\colon A \otimes A \to A$, then the corresponding algebra in $\mathcal{M}^{H_\sigma}$ is A as an H-comodule but with new multiplication:

$$m_\sigma(a \otimes b) = \sigma(a_{(1)}, b_{(1)})m(a_{(0)} \otimes b_{(0)}).$$

We denote this new algebra by A_σ and call it the σ-*twist* of A. It is shown in [23] that multilinear polynomial identities of A are preserved under σ-twist if we interpret them in each of the categories $\mathcal{M}^H$ and $\mathcal{M}^{H_\sigma}$ in terms of the appropriate action of symmetric groups on tensor powers of A. In particular, associative algebras remain associative and (H,β)-Lie algebras become (H_σ,β_σ)-Lie algebras.

If H is cocommutative then $H_\sigma = H$ but β is twisted. If $H = \Bbbk G$, with G an abelian group, then there exists a 2-cocycle $\sigma\colon G\times G\to \Bbbk^\times$ such that β_σ is a "sign bicharacter":

$$\beta_\sigma(g,h) = \begin{cases} -1 & \text{if } g,h\in G_-,\\ 1 & \text{otherwise;}\end{cases}$$

where $G_- = G\setminus G_+$ and G_+ is a subgroup of index ≤ 2. It follows that σ twists any color Lie superalgebra L with commutation factor β into a Lie superalgebra with even part L_+ and odd part L_-, where $L_\pm = \bigoplus_{g\in G_\pm} L_g$.

Etingof and Gelaki [11] showed that, under a certain condition on the antipode called *pseudo-involutivity*, a cotriangular Hopf algebra (H,β) can be twisted by a suitable cocycle to become the algebra of regular functions on a pro-algebraic group G such that $\beta_\sigma = \frac{1}{2}(\varepsilon\otimes\varepsilon+\varepsilon\otimes a+a\otimes\varepsilon-a\otimes a)$ for some central element $a\in G$ with $a^2=1$. It immediately follows [23, Theorem 4.3] that the same cocycle twists (H,β)-Lie algebras to Lie superalgebras equipped with a G-action. Here the even and odd components are just the eigenspaces with respect to the action of a, with eigenvalues 1 and -1 respectively.

If H is finite-dimensional then pseudo-involutivity of the antipode is equivalent to involutivity and hence to semisimplicity of H. Later, Etingof and Gelaki [12, 15] described all finite-dimensional cotriangular Hopf algebras by showing that (H,β) can be twisted in such a way that its dual triangular Hopf algebra becomes a "modified supergroup algebra". As a corollary, any (H,β)-Lie algebra is twisted to a Lie superalgebra equipped with a supergroup action [23, Theorem 4.6].

One can use the twisting procedure to transfer known properties of Lie superalgebras to (H,β)-Lie algebras in the above cases. Let $\mathcal{U}_\beta(L)$ be the universal enveloping algebra of an (H,β)-Lie algebra L, i.e., $\mathcal{U}_c(L)$ for $c=c_{L,L}$ determined by β. It is straightforward to verify that $\mathcal{U}_{\beta_\sigma}(L_\sigma)$ is naturally isomorphic to $(\mathcal{U}_\beta(L))_\sigma$. In particular, for V in $\mathcal{M}^H$ and $c=c_{V,V}$ induced by β, the σ-twist of the Nichols algebra $\mathcal{B}(V,c)$ is naturally isomorphic to $\mathcal{B}(V,c')$ where c' is the braiding on V induced by β_σ. This gives an alternative proof of PBW Theorem for (H,β)-Lie algebras [23].

Theorem 3.3. *Let (H,β) be a cotriangular Hopf algebra that is either pseudo-involutive or finite-dimensional. Let V be a finite-dimensional H-comodule with the corresponding braiding c. If the Nichols algebra $\mathcal{B}(V,c)$ is finite-dimensional then it does not admit nontrivial graded deformations as an augmented algebra or bialgebra in $\mathcal{M}^H$.*

Proof. By our assumption on (H,β), there exists a cocycle σ such that (H_σ,β_σ) is as described by Etingof and Gelaki. Then the braiding c' induced by β_σ on V is just the signed flip associated to a $\mathbb{Z}_2$-grading $V=V_0\oplus V_1$, so $\mathcal{B}(V,c')=S(V_0)\otimes\Lambda(V_1)$, which is finite-dimensional only if $V_0=0$. But in this case V does not admit nontrivial Lie superalgebra structures. It follows that V does not admit nontrivial (H,β)-Lie algebra structures and hence $\mathcal{B}(V,c)$ is rigid in $\mathcal{M}^H$. □

Corollary 3.4. *Let (V,c) be a finite-dimensional braided vector space such that c can be obtained from a coaction by a finite-dimensional cotriangular Hopf algebra. If $\mathcal{B}(V,c)$*

is finite-dimensional then it does not admit nontrivial graded deformations as a braided augmented algebra or bialgebra.

Proof. By assumption, V can be regarded as an object in $\mathcal{M}^H$ for some finite-dimensional cotriangular Hopf algebra (H, β) such that $c = c_{V,V}$. Any graded deformation of $\mathcal{B}(V, c)$ can be realized in $\mathcal{M}^{\bar{H}}$ for some quotient $(\bar{H}, \bar{\beta})$ of the cotriangular Hopf algebra (H, β), so it must be trivial by the above theorem. □

4. The vanishing of second algebra cohomology for a class of augmented algebras in a braided category

Let $\mathcal{V}$ be a braided tensor category consisting of vector spaces and linear maps. Let $(\mathcal{B}, \varepsilon)$ be an augmented algebra in $\mathcal{V}$ acting trivially (i.e., via ε) on some U in $\mathcal{V}$.

⋄ A map $f\colon \mathcal{B}\otimes\mathcal{B} \to U$ in $\mathcal{V}$ is an ε-*cocycle* if $f(1, a) = 0 = f(a, 1)$ and $f(xy, z) = f(x, yz)$ for all $a \in \mathcal{B}$ and all $x, y, z \in \mathcal{B}^+$. The space of all ε-cocycles is denoted by $\mathrm{Z}^2_\varepsilon(\mathcal{B}, U)$.

⋄ An ε-cocycle is an ε-*coboundary* if there exists a map $t\colon \mathcal{B} \to U$ such that $t(1) = 0$ and $f(x, y) = t(xy)$ for all $x, y \in \mathcal{B}^+$. The space of all ε-coboundaries is denoted by $\mathrm{B}^2_\varepsilon(\mathcal{B}, U)$.

⋄ The quotient of ε-cocycles by ε-coboundaries is denoted by $\mathrm{H}^2_\varepsilon(\mathcal{B}, U) = \mathrm{Z}^2_\varepsilon(\mathcal{B}, U)/\mathrm{B}^2_\varepsilon(\mathcal{B}, U)$.

In what follows $(\mathcal{B}^+)^2$ denotes the range of the multiplication $\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \overset{\mathrm{m}}{\to} \mathcal{B}^+$, i.e., $(\mathcal{B}^+)^2 = \operatorname{span}\{xy \mid x, y \in \mathcal{B}^+\}$.

Lemma 4.1 (cf. [25, Subsection 4.1])**.** *Let $\mathcal{B}$ be an augmented algebra in $\mathcal{V}$ and let $M = \ker\left(\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \overset{\mathrm{m}}{\to} \mathcal{B}\right)$. If the map $\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \overset{\mathrm{m}}{\to} (\mathcal{B}^+)^2$ splits in $\mathcal{V}$, then for every space $U \in \mathcal{V}$, we have $\mathrm{H}^2_\varepsilon(\mathcal{B}, U) = \operatorname{Hom}(M, U)$.*

Proof. Let $\varphi\colon (\mathcal{B}^+)^2 \to \mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+$ be a splitting of m and let $p\colon \mathcal{B}^+ \otimes \mathcal{B}^+ \to \mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+$ be the canonical projection. We define a map $\Phi\colon \operatorname{Hom}(M, U) \to \mathrm{H}^2_\varepsilon(\mathcal{B}, U)$ as follows: if $f\colon M \to U$, then the cocycle $\Phi(f)\colon \mathcal{B}^+ \otimes \mathcal{B}^+ \to U$ is $\Phi(f) = f(p - \varphi\,\mathrm{m})$. The inverse Ψ of Φ is defined as follows: if $g\colon \mathcal{B}^+ \otimes \mathcal{B}^+ \to U$ is a cocycle, then $\Psi(g)\colon M \to U$ is the unique map such that $\Psi(g)p = g$. Now observe that maps Φ and Ψ are well defined: $\Phi(f)$ is always a cocycle and $\Psi(g) = 0$ whenever g is a coboundary. Note also that $\Psi\Phi = \mathrm{id}$ and that the range of $\Phi\Psi - \mathrm{id}$ consists of coboundaries. □

Remark 4.2. *A splitting of $\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \overset{\mathrm{m}}{\to} (\mathcal{B}^+)^2$ in $\mathcal{V}$ automatically exists (it is usually not unique) if $\mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+$ is a semisimple object in $\mathcal{V}$. This happens whenever $\mathcal{V}$ is either the category of Yetter-Drinfeld modules over a semisimple and cosemisimple Hopf algebra or the category of comodules over a cosemisimple CQT bialgebra. It also happens if $\mathcal{V}$ is the category of Yetter-Drinfeld modules over $\Bbbk\Gamma$, where Γ is a possibly infinite abelian group, and $\mathcal{B}$ is a direct sum of its one-dimensional subobjects in $\mathcal{V}$ (e.g., a quotient of the tensor algebra $T(V)$, for some V of finite dimension over $\Bbbk$).*

Let V be a an object in $\mathcal{V}$, $T(V)$ its tensor algebra and I an ideal generated by homogeneous elements of degree at least two. Let $\mathcal{B} = T(V)/I$ and let $\pi\colon T(V) \to \mathcal{B}$ be the canonical projection. We also abbreviate $T(V)^+ = \bigoplus_{n\geq 1} V^{\otimes n}$ and $T(V)_{(2)} = \bigoplus_{n\geq 2} V^{\otimes n}$.

Lemma 4.3. *The following is a commutative diagram:*

$$
\begin{array}{ccccccc}
 & & & & I\otimes T(V)^+ + T(V)^+\otimes I & \xrightarrow{\mathrm{m}} & I \\
 & & & & \big\downarrow & & \big\downarrow \\
T(V)^+\otimes T(V)\otimes T(V)^+ & & \xrightarrow{\mathrm{id}\otimes\mathrm{m}-\mathrm{m}\otimes\mathrm{id}} & & T(V)^+\otimes T(V)^+ & \xrightarrow{\mathrm{m}} & T(V)_{(2)} \\
\big\downarrow{\scriptstyle \pi\otimes\pi\otimes\pi} & & & & \big\downarrow{\scriptstyle \pi\otimes\pi} & & \big\downarrow{\scriptstyle \widetilde{\pi\otimes\pi}} \\
\mathcal{B}^+\otimes\mathcal{B}\otimes\mathcal{B}^+ & & \xrightarrow{\mathrm{id}\otimes\mathrm{m}-\mathrm{m}\otimes\mathrm{id}} & & \mathcal{B}^+\otimes\mathcal{B}^+ & \xrightarrow{p} & \mathcal{B}^+\otimes_{\mathcal{B}}\mathcal{B}^+ \\
 & & & & \big\downarrow{\scriptstyle \mathrm{m}} & & \big\downarrow{\scriptstyle \widetilde{\mathrm{m}}} \\
 & & & & (\mathcal{B}^+)^2 & = & (\mathcal{B}^+)^2
\end{array}
$$

where the maps $\widetilde{\mathrm{m}}$ *and* $\widetilde{\pi\otimes\pi}$ *are the universal maps arising from fact (1) below. Moreover, we have the following facts:*

(1) *The second and third rows of the diagram are cokernel diagrams.*
(2) *The second column of the diagram is exact at* $T(V)^+\otimes T(V)^+$.
(3) *The composition* $T(V)_{(2)} \overset{\widetilde{\pi\otimes\pi}}{\to} \mathcal{B}^+\otimes_{\mathcal{B}}\mathcal{B}^+ \overset{\widetilde{\mathrm{m}}}{\to} (\mathcal{B}^+)^2$ *is equal to the restriction of* π *to* $T(V)_{(2)}$.
(4) *The map* $\widetilde{\pi\otimes\pi}$ *is surjective.*
(5) *If* $\varphi\colon T(V)_{(2)}\to T(V)^+\otimes T(V)^+$ *is any splitting of multiplication (e.g., the composition* $T(V)_{(2)} \overset{\sim}{\to} V\otimes T(V)^+ \to T(V)^+\otimes T(V)^+$ *is such a splitting), then* $\widetilde{\pi\otimes\pi} = p(\pi\otimes\pi)\varphi$.

Proof. Clearly, each of the squares of the diagram commutes. We prove the remaining claims below:

(1) The third row is a cokernel diagram by definition. The second row is a cokernel diagram due to the fact that $T(V)^+ = V\otimes T(V)$ as a right $T(V)$-module (with the obvious action on the second tensor factor), hence $T(V)^+\otimes_{T(V)}T(V)^+ = V\otimes T(V)^+$, and $V\otimes T(V)^+ \overset{\mathrm{m}}{\to} T(V)_{(2)}$ is an isomorphism.
(2) Clear.
(3) As π is an algebra map, we have $\mathrm{m}(\pi\otimes\pi)\,\mathrm{m} = \pi\,\mathrm{m}$. Hence $\widetilde{\mathrm{m}}(\widetilde{\pi\otimes\pi})\,\mathrm{m} = \pi\,\mathrm{m}$. By the universal property of cokernels this means that $\widetilde{\mathrm{m}}(\widetilde{\pi\otimes\pi}) = \pi$.
(4) Follows from the fact that maps p and $\pi\otimes\pi$ are surjective.
(5) Follows from the universal property of cokernels.

□

Corollary 4.4. *The following sequence is exact:*

$$0\to T(V)^+I + IT(V)^+ \to I \xrightarrow{\widetilde{\pi\otimes\pi}} \mathcal{B}^+\otimes_{\mathcal{B}}\mathcal{B}^+ \overset{\widetilde{\mathrm{m}}}{\to} (\mathcal{B}^+)^2 \to 0$$

Therefore, $I/(T(V)^+I + IT(V)^+) \simeq \ker\big(\mathcal{B}^+\otimes_{\mathcal{B}}\mathcal{B}^+ \to (\mathcal{B}^+)^2\big)$.

Proof. To avoid ambiguity, we denote the restriction of $\widetilde{\pi\otimes\pi}$ to I by τ. We first prove that $\ker(\tau) = T(V)^+I + IT(V)^+$. The inclusion $T(V)^+I + IT(V)^+ \subseteq \ker(\widetilde{\pi\otimes\pi})$ follows from

$\widetilde{\pi\otimes\pi}(T(V)^+I + IT(V)^+) = (\widetilde{\pi\otimes\pi})\,\mathrm{m}(T(V)^+\otimes I + I\otimes T(V)^+) = p(\pi\otimes\pi)(T(V)^+\otimes I + I\otimes T(V)^+) = 0$.

Let $x \in \ker(\tau)$. Since $\mathrm{m}(T(V)^+\otimes T(V)^+) = T(V)_{(2)}$, there exists $y \in T(V)^+\otimes T(V)^+$ such that $\mathrm{m}(y) = x$. Now $0 = (\widetilde{\pi\otimes\pi})\,\mathrm{m}(y) = p(\pi\otimes\pi)(y)$, and hence $(\pi\otimes\pi)y = (\mathrm{id}\otimes\mathrm{m} - \mathrm{m}\otimes\mathrm{id})z$ for some $z \in \mathcal{B}^+\otimes\mathcal{B}\otimes\mathcal{B}^+$. Let $w \in T(V)^+\otimes T(V)\otimes T(V)^+$ be such that $(\pi\otimes\pi\otimes\pi)(w) = z$. Define $y' = y - (\mathrm{id}\otimes\mathrm{m} - \mathrm{m}\otimes\mathrm{id})w$. As $(\pi\otimes\pi)y' = 0$ we have that $y' \in I\otimes T(V)^+ + T(V)^+\otimes I$ and hence $x = \mathrm{m}(y) = \mathrm{m}(y') \in IT(V)^+ + T(V)^+I$.

We now prove that $\widetilde{\pi\otimes\pi}(I) = \ker(\mathcal{B}^+\otimes_{\mathcal{B}}\mathcal{B}^+ \xrightarrow{\widetilde{\mathrm{m}}} (\mathcal{B}^+)^2)$. The inclusion $\subseteq$ follows from part (3) of the lemma above: $\widetilde{\mathrm{m}}(\widetilde{\pi\otimes\pi})(I) = \pi(I) = 0$. The inclusion $\supseteq$ follows from the fact that $\widetilde{\pi\otimes\pi}$ is surjective. □

Corollary 4.5. *If I is generated by a subobject R, then the induced morphism*

$$R \to \ker\left(\mathcal{B}^+\otimes_{\mathcal{B}}\mathcal{B}^+ \to (\mathcal{B}^+)^2\right)$$

is surjective. □

We summarize the above results in a theorem which will be needed in the next section to establish rigidity of certain graded bialgebras in $\mathcal{V}$.

Theorem 4.6. *Let V be a an object in $\mathcal{V}$ and $T(V)$ its tensor algebra. Let $R \subset T(V)_{(2)}$ be a graded subspace that is an object in $\mathcal{V}$. Consider the augmented algebra $\mathcal{B} = T(V)/\langle R\rangle$ and an object U in $\mathcal{V}$ on which $\mathcal{B}$ acts trivially (i.e., via ε). If the multiplication map $\mathcal{B}^+\otimes_{\mathcal{B}}\mathcal{B}^+ \xrightarrow{\mathrm{m}} (\mathcal{B}^+)^2$ splits in $\mathcal{V}$, then there is an injection $\mathrm{H}^2_\varepsilon(\mathcal{B}, U) \to \mathrm{Hom}(R, U)$.*

In particular, if f is an ε-cocycle such that for every $u \in \mathcal{B}\otimes\mathcal{B}$ in the range of the composition $R \to V\otimes T(V)^+ \to \mathcal{B}\otimes\mathcal{B}$ we have $f(u) = 0$, then f is an ε-coboundary. □

5. A sufficient condition for rigidity of graded bialgebras in a braided category

Let $\mathcal{B}$ be a graded bialgebra in $\mathcal{V}$. For a homogeneous map $f\colon \mathcal{B}\otimes\mathcal{B} \to \mathcal{B}$ of degree ℓ and a nonnegative integer r we define $f_r\colon \mathcal{B}\otimes\mathcal{B} \to \mathcal{B}$ by $f_r|_{(\mathcal{B}\otimes\mathcal{B})_r} = f$ and $f_r|_{(\mathcal{B}\otimes\mathcal{B})_s} = 0$ for $s \neq r$. For $g\colon \mathcal{B} \to \mathcal{B}\otimes\mathcal{B}$, we define g_r analogously. We also define $f_{\le r}$ by $f_{\le r} = \sum_{i=0}^r f_i$, and $f_{<r}, g_{\le r}, g_{<r}$ in a similar fashion.

Lemma 5.1 (cf. [25, Lemma 2.3.6])**.** *Let $\mathcal{B}$ be a graded bialgebra in $\mathcal{V}$ such that $\mathcal{B}_0 = \Bbbk$ and $\mathcal{B}$ is generated as an algebra by $\mathcal{B}_1$.*

1. *If $(f, g) \in \mathrm{Z}^2_{\mathrm{b}}(\mathcal{B})_\ell$, $r > 1$, $f_{\le r} = 0$, and $g_{<r} = 0$, then $g_r = 0$.*
2. *If $(f, g) \in \mathrm{Z}^2_{\mathrm{b}}(\mathcal{B})_\ell$, $\ell < 0$, and $f_{\le r} = 0$, then $g_{\le r} = 0$.*
3. *If $(0, g) \in \mathrm{Z}^2_{\mathrm{b}}(\mathcal{B})_\ell$, $\ell < 0$, then $g = 0$.*

Proof. The proof in [25] carries over word for word. First note that for every $(f, g) \in \mathrm{Z}^2_{\mathrm{b}}(\mathcal{B})$ we have $f_{\le 1} = 0$ and $g_{\le 2} = 0$, due to the fact that $(\mathcal{B}^+\otimes\mathcal{B}^+)_0 = 0 = (\mathcal{B}^+\otimes\mathcal{B}^+)_1$. Hence (1) easily yields (2) and (3).

For (1) recall that $\partial^c f = -\partial^h g$ by Equation (4). If $r > 1$, $a \in \mathcal{B}_1$ and $b \in \mathcal{B}_{r-1}$, then

$$(\partial^c f)(a, b) = 0 = -(\partial^h g)(a, b) = -(\Delta a)g(b) + g(ab) - g(a)(\Delta b) = g(ab).$$

As $\mathcal{B}_r$ is spanned by such products ab, we have that $g(\mathcal{B}_r) = 0$. □

Lemma 5.2 (cf. [25, Lemma 2.3.5]). *Let $\mathcal{B}$ be a connected graded bialgebra in $\mathcal{V}$, let $r \in \mathbb{N}$, and let $f\colon \mathcal{B}\otimes\mathcal{B} \to \mathcal{B}$ be a homogeneous unital Hochschild cocycle in $\mathcal{V}$ (with respect to left and right regular actions of $\mathcal{B}$ on itself). If $f_{<r} = 0$, then $f_r\colon \mathcal{B}\otimes\mathcal{B}\to\mathcal{B}$ is an ε-cocycle.*

Proof. This follows directly from $\partial^h f = 0$, see Equation (2). □

Theorem 5.3 (cf. [25, Lemma 4.2.2]). *Let V be a an object in $\mathcal{V}$ and $T(V)$ its (braided) tensor bialgebra. Let $R \subset T(V)_{(2)}$ be a graded subspace that is an object in $\mathcal{V}$ and generates a biideal in $T(V)$. Consider the quotient $\mathcal{B} = T(V)/\langle R\rangle$, which is a graded bialgebra in $\mathcal{V}$, and assume that the multiplication map $\mathrm{m}\colon \mathcal{B}^+ \otimes_{\mathcal{B}} \mathcal{B}^+ \to (\mathcal{B}^+)^2$ splits in $\mathcal{V}$. If for some negative ℓ we have that $\mathrm{Hom}(R, P(\mathcal{B}))_\ell = 0$, then $\widehat{\mathrm{H}}^2_{\mathrm{b}}(\mathcal{B})_\ell = 0$.*

In particular, if $\mathrm{Hom}(R, P(\mathcal{B}))_\ell = 0$ for all negative ℓ, then $\mathcal{B}$ is rigid.

Proof. Let $(f, g) \in \mathrm{Z}^2_{\mathrm{b}}(\mathcal{B})_\ell$. We will find a map $s = \sum_{r=0}^{\infty} s_r\colon \mathcal{B}\to\mathcal{B}$ such that for every nonnegative r, $(f,g)_r = (\partial^h s_r, -\partial^c s_r)$, from where the result trivially follows since $(f, g) = \partial^b \sum_{r=0}^{\infty} s_r$. Here the sum $s = \sum_{r=0}^{\infty} s_r$ is potentially infinite but locally finite. The cases $r = 0, 1$ are clear. Suppose that $s_0, \dots s_{r-1}$ have been found. Let $(f', g') = (f, g) - \partial^b s_{<r} = (f, g) - \sum_{i=0}^{r-1}(\partial^h s_i, -\partial^c s_i)$. Note that, by assumption, $f'|_{<r} = 0$ and hence, by Lemma 5.1, also $g'|_{<r} = 0$. Let $u \in (\mathcal{B}\otimes\mathcal{B})$ be in the range of the composition $R \to V\otimes T(V)^+ \to \mathcal{B}\otimes\mathcal{B}$. Since $\mathrm{m}(u) = 0$ we have from Equation (4) that $f_r(u) \in P(\mathcal{B})$. Therefore, the composition $R \to V \otimes T(V)^+ \to \mathcal{B}\otimes\mathcal{B} \xrightarrow{f} \mathcal{B}$ has range in $P(\mathcal{B})$ and must be the zero map. By Theorem 4.6, we get a map $t\colon \mathcal{B}\to\mathcal{B}$ such that $f_r = t\,\mathrm{m}$. Now define $s_r = t_r$ and observe that $f'_{\le r} = f'_r = \partial^h s_r$. Hence, by Lemma 5.1, we also have $g'_r = -\partial^c s_r$. □

6. Nichols algebras of diagonal type

In what follows (V, c) will denote a braided vector space of diagonal type, $\dim V = \theta$, such that the associated Nichols algebra $\mathcal{B}(V)$ has a finite root system Δ^V_+ in the sense of [19], i.e., Δ^V_+ is the set of $\mathbb{N}_0^\theta$-degrees of generators of a PBW basis. In particular, this is the case if $\mathcal{B}(V)$ is finite-dimensional. Let

$$-c^V_{ij} := \min\left\{n \in \mathbb{N}_0 \mid (n+1)_{q_{ii}}(1 - q_{ii}^n q_{ij} q_{ji}) = 0\right\}, \qquad j \neq i. \tag{5}$$

Now we fix

- a basis $\{x_1, \dots, x_\theta\}$ of V and $q_{ij} \in \Bbbk^\times$ such that $c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$,
- elements $x_\alpha \in \mathcal{B}(V)$ of degree α, $\alpha \in \Delta^V_+$, which generate a PBW basis, see [4].

We use the following notation:

- $\widetilde{q_{ij}} := q_{ij}q_{ji}$ for all $i \neq j$.
- $\chi\colon \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \Bbbk^\times$ is the bicharacter such that $\chi(\alpha_i, \alpha_j) = q_{ij}$, $1 \le i, j \le \theta$, where $\{\alpha_1, \dots, \alpha_\theta\}$ is the canonical basis of $\mathbb{Z}^\theta$.
- N_α is the order of $q_\alpha := \chi(\alpha, \alpha)$, $\alpha \in \Delta^V_+$.
- $\mathbb{G}_N$ is the group of roots of unity of order N and $\mathbb{G}'_N$ is the subset of primitive roots of unity of order N, $N \in \mathbb{N}$.
- $\mathcal{O}(V)$ is the set of Cartan roots of V, i.e., the orbit of Cartan vertices under the action of the Weyl groupoid. Recall that $i \in \{1, \dots, \theta\}$ is a Cartan vertex of V if $\widetilde{q_{ij}} = q_{ii}^{c^V_{ij}}$ for all $j \neq i$ [4, Definition 2.6].

We recall the following result, which gives a presentation by generators and relations for any Nichols algebra of diagonal type with finite root system.

Theorem 6.1. [4] *$\mathcal{B}(V)$ is presented by generators $x_1, \dots, x_\theta$ and relations:*

(6) $$x_\alpha^{N_\alpha}, \qquad \alpha \in \mathcal{O}(V);$$

(7) $$(\operatorname{ad}_c x_i)^{1-c_{ij}^V} x_j, \qquad q_{ii}^{1-c_{ij}^V} \neq 1;$$

(8) $$x_i^{N_i}, \qquad i \text{ is not a Cartan vertex};$$

◇ *if $i, j \in \{1, \dots, \theta\}$ satisfy $q_{ii} = \widetilde{q_{ij}} = q_{jj} = -1$, and there exists $k \neq i, j$ such that $\widetilde{q_{ik}}^2 \neq 1$ or $\widetilde{q_{jk}}^2 \neq 1$,*

(9) $$x_{ij}^2;$$

◇ *if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{jj} = -1$, $\widetilde{q_{ik}} = \widetilde{q_{ij}}\widetilde{q_{kj}} = 1$, $\widetilde{q_{ij}} \neq -1$,*

(10) $$[x_{ijk}, x_j]_c;$$

◇ *if $i, j \in \{1, \dots, \theta\}$ satisfy $q_{jj} = -1$, $q_{ii}\widetilde{q_{ij}} \in \mathbb{G}_6'$, $\widetilde{q_{ij}} \neq -1$, and also $q_{ii} \in \mathbb{G}_3'$ or $-c_{ij}^V \geq 3$,*

(11) $$[x_{iij}, x_{ij}]_c;$$

◇ *if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{ii} = \pm\widetilde{q_{ij}} \in \mathbb{G}_3'$, $\widetilde{q_{ik}} = 1$, and also $-q_{jj} = \widetilde{q_{ij}}\widetilde{q_{jk}} = 1$ or $q_{jj}^{-1} = \widetilde{q_{ij}} = \widetilde{q_{jk}} \neq -1$,*

(12) $$[x_{iijk}, x_{ij}]_c;$$

◇ *if $i, j, k \in \{1, \dots, \theta\}$ satisfy $\widetilde{q_{ik}}, \widetilde{q_{ij}}, \widetilde{q_{jk}} \neq 1$,*

(13) $$x_{ijk} - \frac{1 - \widetilde{q_{jk}}}{q_{kj}(1 - \widetilde{q_{ik}})} [x_{ik}, x_j]_c - q_{ij}(1 - \widetilde{q_{jk}})\, x_j x_{ik};$$

◇ *if $i, j, k \in \{1, \dots, \theta\}$ satisfy one of the following situations*
- ***(i)*** *$q_{ii} = q_{jj} = -1$, $\widetilde{q_{ij}}^2 = \widetilde{q_{jk}}^{-1}$, $\widetilde{q_{ik}} = 1$, or*
- ***(ii)*** *$\widetilde{q_{ij}} = q_{jj} = -1$, $q_{ii} = -\widetilde{q_{jk}}^2 \in \mathbb{G}_3'$, $\widetilde{q_{ik}} = 1$, or*
- ***(iii)*** *$q_{kk} = \widetilde{q_{jk}} = q_{jj} = -1$, $q_{ii} = -\widetilde{q_{ij}} \in \mathbb{G}_3'$, $\widetilde{q_{ik}} = 1$, or*
- ***(iv)*** *$q_{jj} = -1$, $\widetilde{q_{ij}} = q_{ii}^{-2}$, $\widetilde{q_{jk}} = -q_{ii}^3$, $\widetilde{q_{ik}} = 1$, or*
- ***(v)*** *$q_{ii} = q_{jj} = q_{kk} = -1$, $\pm\widetilde{q_{ij}} = \widetilde{q_{jk}} \in \mathbb{G}_3'$, $\widetilde{q_{ik}} = 1$,*

(14) $$\big[[x_{ij}, x_{ijk}]_c, x_j\big]_c;$$

◇ *if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{ii} = q_{jj} = -1$, $\widetilde{q_{ij}}^3 = \widetilde{q_{jk}}^{-1}$, $\widetilde{q_{ik}} = 1$,*

(15) $$\big[[x_{ij}, [x_{ij}, x_{ijk}]_c]_c, x_j\big]_c;$$

◇ *if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{jj} = \widetilde{q_{ij}}^2 = \widetilde{q_{jk}} \in \mathbb{G}_3'$, $\widetilde{q_{ik}} = 1$,*

(16) $$\big[[x_{ijk}, x_j]_c x_j\big]_c;$$

◇ *if $i, j, k \in \{1, \dots, \theta\}$ satisfy $q_{kk} = q_{jj} = \widetilde{q_{ij}}^{-1} = \widetilde{q_{jk}}^{-1} \in \mathbb{G}_9'$, $\widetilde{q_{ik}} = 1$, $q_{ii} = q_{kk}^6$*

(17) $$\big[[x_{iij}, x_{iijk}]_c, x_{ij}\big]_c;$$

⋄ *if* $i,j,k \in \{1,\dots,\theta\}$ *satisfy* $q_{ii} = \widetilde{q_{ij}}^{-1} \in \mathbb{G}'_9$, $q_{jj} = \widetilde{q_{jk}}^{-1} = q_{ii}^5$, $\widetilde{q_{ik}} = 1$, $q_{kk} = q_{ii}^6$

$$[[x_{ijk}, x_j]_c, x_k]_c - (1+\widetilde{q_{jk}})^{-1} q_{jk} \left[[x_{ijk}, x_k]_c, x_j\right]_c; \tag{18}$$

⋄ *if* $i,j,k \in \{1,\dots,\theta\}$ *satisfy* $q_{jj} = \widetilde{q_{ij}}^3 = \widetilde{q_{jk}} \in \mathbb{G}'_4$, $\widetilde{q_{ik}} = 1$,

$$\left[\left[[x_{ijk}, x_j]_c, x_j\right]_c, x_j\right]_c; \tag{19}$$

⋄ *if* $i,j,k \in \{1,\dots,\theta\}$ *satisfy* $q_{ii} = \widetilde{q_{ij}} = -1$, $q_{jj} = \widetilde{q_{jk}}^{-1} \neq -1$, $\widetilde{q_{ik}} = 1$,

$$[x_{ij}, x_{ijk}]_c; \tag{20}$$

⋄ *if* $i,j,k \in \{1,\dots,\theta\}$ *satisfy* $q_{ii} = q_{kk} = -1$, $\widetilde{q_{ik}} = 1$, $\widetilde{q_{ij}} \in \mathbb{G}'_3$, $q_{jj} = -\widetilde{q_{jk}} = \pm\widetilde{q_{ij}}$,

$$[x_i, x_{jjk}]_c - (1+q_{jj}^2) q_{kj}^{-1} [x_{ijk}, x_j]_c - (1+q_{jj}^2)(1+q_{jj}) q_{ij} x_j x_{ijk}; \tag{21}$$

⋄ *if* $i,j,k,l \in \{1,\dots,\theta\}$ *satisfy* $q_{jj}\widetilde{q_{ij}} = q_{jj}\widetilde{q_{jk}} = 1$, $q_{kk} = -1$, $\widetilde{q_{ik}} = \widetilde{q_{il}} = \widetilde{q_{jl}} = 1$, $\widetilde{q_{jk}}^2 = \widetilde{q_{lk}}^{-1} = q_{ll}$,

$$\left[\left[[x_{ijkl}, x_k]_c, x_j\right]_c, x_k\right]_c; \tag{22}$$

⋄ *if* $i,j,k,l \in \{1,\dots,\theta\}$ *satisfy* $\widetilde{q_{jk}} = \widetilde{q_{ij}} = q_{jj}^{-1} \in \mathbb{G}'_4 \cup \mathbb{G}'_6$, $q_{ii} = q_{kk} = -1$, $\widetilde{q_{ik}} = \widetilde{q_{il}} = \widetilde{q_{jl}} = 1$, $\widetilde{q_{jk}}^3 = \widetilde{q_{lk}}$,

$$\left[\left[x_{ijk}, [x_{ijkl}, x_k]_c\right]_c, x_{jk}\right]_c; \tag{23}$$

⋄ *if* $i,j,k,l \in \{1,\dots,\theta\}$ *satisfy* $q_{ll} = \widetilde{q_{lk}}^{-1} = q_{kk} = \widetilde{q_{jk}}^{-1} = q^2$, $\widetilde{q_{ij}} = q_{ii}^{-1} = q^3$ *for some* $q \in \Bbbk^\times$, $q_{jj} = -1$, $\widetilde{q_{ik}} = \widetilde{q_{il}} = \widetilde{q_{jl}} = 1$,

$$\left[\left[[x_{ijk}, x_j]_c, [x_{ijkl}, x_j]_c\right]_c, x_{jk}\right]_c; \tag{24}$$

⋄ *if* $i,j,k,l \in \{1,\dots,\theta\}$ *satisfy one of the following situations*
 (i) $q_{kk} = -1$, $q_{ii} = \widetilde{q_{ij}}^{-1} = q_{jj}^2$, $\widetilde{q_{kl}} = q_{ll}^{-1} = q_{jj}^3$, $\widetilde{q_{jk}} = q_{jj}^{-1}$, $\widetilde{q_{ik}} = \widetilde{q_{il}} = \widetilde{q_{jl}} = 1$, *or*
 (ii) $q_{ii} = \widetilde{q_{ij}}^{-1} = -q_{ll}^{-1} = -\widetilde{q_{kl}}$, $q_{jj} = \widetilde{q_{jk}} = q_{kk} = -1$, $\widetilde{q_{ik}} = \widetilde{q_{il}} = \widetilde{q_{jl}} = 1$,

$$\left[[x_{ijkl}, x_j]_c, x_k\right]_c - q_{jk}(\widetilde{q_{ij}}^{-1} - q_{jj}) \left[[x_{ijkl}, x_k]_c, x_j\right]_c; \tag{25}$$

⋄ *if* $i,j,k \in \{1,\dots,\theta\}$ *satisfy* $\widetilde{q_{jk}} = 1$, $q_{ii} = \widetilde{q_{ij}} = -\widetilde{q_{ik}} \in \mathbb{G}'_3$,

$$\left[x_i, [x_{ij}, x_{ik}]_c\right]_c + q_{jk} q_{ik} q_{ji} [x_{iik}, x_{ij}]_c + q_{ij}\, x_{ij} x_{iik}; \tag{26}$$

⋄ *if* $i,j,k \in \{1,\dots,\theta\}$ *satisfy* $q_{jj} = q_{kk} = \widetilde{q_{jk}} = -1$, $q_{ii} = -\widetilde{q_{ij}} \in \mathbb{G}'_3$, $\widetilde{q_{ik}} = 1$,

$$[x_{iijk}, x_{ijk}]_c; \tag{27}$$

⋄ *if* $i,j \in \{1,\dots,\theta\}$ *satisfy* $-q_{ii}, -q_{jj}, q_{ii}\widetilde{q_{ij}}, q_{jj}\widetilde{q_{ij}} \neq 1$,

$$(1-\widetilde{q_{ij}}) q_{jj} q_{ji} \left[x_i, [x_{ij}, x_j]_c\right]_c - (1+q_{jj})(1-q_{jj}\widetilde{q_{ij}}) x_{ij}^2; \tag{28}$$

⋄ *if* $i,j \in \{1,\dots,\theta\}$ *satisfy that* $-c_{ij}^V \in \{4,5\}$, *or* $q_{jj} = -1$, $-c_{ij}^V = 3$, $q_{ii} \in \mathbb{G}'_4$,

$$\left[x_i, x_{3\alpha_i+2\alpha_j}\right]_c - \frac{1 - q_{ii}\widetilde{q_{ij}} - q_{ii}^2 \widetilde{q_{ij}}^2 q_{jj}}{(1-q_{ii}\widetilde{q_{ij}}) q_{ji}} x_{iij}^2; \tag{29}$$

◇ *if* $i, j \in \{1, \dots, \theta\}$ *satisfy* $4\alpha_i + 3\alpha_j \notin \Delta_+^V$, $q_{jj} = -1$ *or* $m_{ji} \geq 2$, *and also* $-c_{ij}^V \geq 3$, *or* $-c_{ij}^V = 2$, $q_{ii} \in \mathbb{G}_3'$,

$$x_{4\alpha_i+3\alpha_j} = [x_{3\alpha_i+2\alpha_j}, x_{ij}]_c; \tag{30}$$

◇ *if* $i, j \in \{1, \dots, \theta\}$ *satisfy* $3\alpha_i + 2\alpha_j \in \Delta_+^V$, $5\alpha_i + 3\alpha_j \notin \Delta_+^V$, *and* $q_{ii}^3\widetilde{q_{ij}}, q_{ii}^4\widetilde{q_{ij}} \neq 1$,

$$[x_{iij}, x_{3\alpha_i+2\alpha_j}]_c; \tag{31}$$

◇ *if* $i, j \in \{1, \dots, \theta\}$ *satisfy* $4\alpha_i + 3\alpha_j \in \Delta_+^V$, $5\alpha_i + 4\alpha_j \notin \Delta_+^V$,

$$x_{5\alpha_i+4\alpha_j} = [x_{4\alpha_i+3\alpha_j}, x_{ij}]_c; \tag{32}$$

◇ *if* $i, j \in \{1, \dots, \theta\}$ *satisfy* $5\alpha_i + 2\alpha_j \in \Delta_+^V$, $7\alpha_i + 3\alpha_j \notin \Delta_+^V$,

$$[[x_{iiij}, x_{iij}], x_{iij}]_c; \tag{33}$$

◇ *if* $i, j \in \{1, \dots, \theta\}$ *satisfy* $q_{jj} = -1$, $5\alpha_i + 4\alpha_j \in \Delta_+^V$,

$$[x_{iij}, x_{4\alpha_i+3\alpha_j}]_c - \frac{b - (1+q_{ii})(1-q_{ii}\zeta)(1+\zeta+q_{ii}\zeta^2)q_{ii}^6\zeta^4}{a\ q_{ii}^3 q_{ij}^2 q_{ji}^3} x_{3\alpha_i+2\alpha_j}^2, \tag{34}$$

where $\zeta = \widetilde{q_{ij}}$, $a = (1-\zeta)(1-q_{ii}^4\zeta^3) - (1-q_{ii}\zeta)(1+q_{ii})q_{ii}\zeta$, $b = (1-\zeta)(1-q_{ii}^6\zeta^5) - a\ q_{ii}\zeta$. □

We fix a realization of (V, c) as a Yetter-Drinfeld module over an abelian group Γ, i.e., there exist $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$ such that $\chi_j(g_i) = q_{ij}$ and we make V an object of ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ by declaring $x_i \in V_{g_i}^{\chi_i}$. Let $\mathcal{R}_V$ be the set of relations defining $\mathcal{B}(V)$ according to the previous theorem. Note that $\Bbbk\mathcal{R}_V$ is a Yetter-Drinfeld submodule of $T(V)$, because each relation is $\mathbb{Z}^\theta$-homogeneous. For each $R \in \mathcal{R}_V$ of degree $(a_1, \dots, a_\theta) \in \mathbb{Z}^\theta$, set

$$g_R := g_1^{a_1} \cdots g_\theta^{a_\theta}, \quad \chi_R := \chi_1^{a_1} \cdots \chi_\theta^{a_\theta}, \qquad \text{so } R \in T(V)_{g_R}^{\chi_R}. \tag{35}$$

The *support* of $R \in \mathcal{R}_V$ is the set $\operatorname{supp} R := \{i \mid a_i \neq 0\}$, i.e., the set of indices of letters x_i appearing in R.

Proposition 6.2. *For every* $R \in \mathcal{R}_V$ *and* $t \in \{1, 2, \dots, \theta\}$, *we have* $(g_R, \chi_R) \neq (g_t, \chi_t)$.

Proof. We prove this for each defining relation. For (7), see [6, Proposition 3.1]; the proof does not use that the braiding is of standard type.

We discard easily the cases (6), (8), (9), (14)**(v)** , (25)**(ii)** , (27), (34) because $\chi_R(g_R) = 1$.

For the remaining cases, note that the propositions in [4, Section 3] show that $(g_R, \chi_R) \neq (g_t, \chi_t)$ for each $t \notin \operatorname{supp} R$. Therefore, we have to consider only the case $t \in \operatorname{supp}(R)$.

For each remaining relation R, we compute $\chi_R(g_R)$ and/or $\{\chi_R(g_t)\chi_t(g_R) \mid t \in \operatorname{supp} R\}$.

(10): we have $\chi_R(g_R) = q_{ii}q_{kk} \neq q_{ii}, q_{kk}$. Suppose that $g_R = g_j$, $\chi_R = \chi_j$. Then $\widetilde{q_{ij}} = \chi_R(g_i)\chi_i(g_R) = (q_{ii}\widetilde{q_{ij}})^2$ and $\widetilde{q_{kj}} = \chi_R(g_k)\chi_k(g_R) = (q_{kk}\widetilde{q_{kj}})^2$, so $q_{ii}^2\widetilde{q_{ij}} = q_{kk}^2\widetilde{q_{kj}} = 1$. But such a generalized Dynkin diagram is not in [19], a contradiction.

(11): now $\chi_R(g_R) = q_{ii}^3 \neq q_{ii}$, $\widetilde{q_{ij}} \neq \chi_R(g_i)\chi_i(g_R) = \widetilde{q_{ij}}^2$, so $(g_R, \chi_R) \neq (g_i, \chi_i), (g_j, \chi_j)$.

(12): for both sets of conditions, $\widetilde{q_{ij}}q_{jj}^2\widetilde{q_{jk}} = 1$ so $\chi_R(g_R) = q_{ii}q_{kk} \neq q_{ii}, q_{kk}$. Suppose that $g_R = g_j$, $\chi_R = \chi_j$. But $\widetilde{q_{ij}} \neq \chi_R(g_i)\chi_i(g_R) = \widetilde{q_{ij}}^2$, a contradiction.

(13): recall that $\widetilde{q_{ij}}\widetilde{q_{ik}}\widetilde{q_{jk}} = 1$. Suppose that $g_R = g_i$, $\chi_R = \chi_i$. Then $q_{ii} = \chi_R(g_i) = q_{ii}q_{ij}q_{ik}$, so $q_{ij}q_{ik} = 1$. Also $q_{ji}q_{ki} = 1$, so $\widetilde{q_{ij}}\widetilde{q_{ik}} = 1$ and then $\widetilde{q_{jk}} = 1$, a contradiction.

(14)**(i)** : simply note that $\chi_R(g_R) = -q_{kk} \neq -1, q_{kk}$.

(14)**(ii)** : as $\chi_R(g_R) = q_{ii}q_{kk} \neq q_{ii}, q_{kk}$, the remaining case is $t = j$. But also $\widetilde{q_{ij}} = -1 \neq \chi_R(g_i)\chi_i(g_R) = -q_{ii}$.

(14)**(iii)** : it follows since $\chi_R(g_R) = -q_{ii} \neq -1, q_{ii}$.

(14)**(iv)** again $\chi_R(g_R) = q_{ii}q_{kk} \neq q_{ii}, q_{kk}$, so the remaining case is $t = j$. Suppose that $g_R = g_j$, $\chi_R = \chi_j$, so $1 = q_{jj}^2 = \chi_R(g_j)\chi_j(g_R) = \widetilde{q_{ij}}^2\widetilde{q_{jk}} = -q_{ii}$, a contradiction.

(15): it follows since $\chi_R(g_R) = -q_{kk} \neq -1, q_{kk}$.

(16): again $\chi_R(g_R) = q_{ii}q_{kk} \neq q_{ii}, q_{kk}$. Suppose that $g_R = g_j$, $\chi_R = \chi_j$, so

$$q_{jj} = q_{ii}q_{kk}, \qquad 1 = \widetilde{q_{ij}}\widetilde{q_{jk}} = \chi_R(g_i)\chi_i(g_R)\chi_R(g_k)\chi_k(g_R) = q_{ii}^2q_{kk}^2 = q_{jj}^2,$$

which is a contradiction.

(17): it follows from $\chi_R(g_R) = q_{jj}^{-2} \neq q_{ii}, q_{jj}, q_{kk}$.

(18): it follows from $\chi_R(g_R) = q_{jj} \neq q_{ii}, q_{kk}$, and $\chi_R(g_i)\chi_i(g_R) = 1 \neq \widetilde{q_{ij}}$.

(19): the proof is analogous to the one for (16).

(20): As $\chi_R(g_R) = q_{jj}^2q_{kk}$ and $q_{jj} \neq \pm 1$, we discard the case $t = k$. The case $t = j$ is also discarded because $1 = \chi_R(g_i)\chi_i(g_R) \neq \widetilde{q_{ij}}$. Finally suppose that $\chi_R = \chi_i$, $g_R = g_i$, so $-1 = \widetilde{q_{ij}} = \chi_R(g_j)\chi_j(g_R) = q_{jj}^3$. Then $q_{jj} \in \mathbb{G}_6'$ and $-1 = \chi_R(g_R) = q_{jj}^2q_{kk}$, so $q_{kk} = q_{jj}$. But this case corresponds to a diagram which is not in [19], a contradiction.

(21): Note that $\chi_R(g_R) = q_{jj}^2 \neq q_{jj}, -1 = q_{ii} = q_{kk}$ because $q_{jj}^2 = \widetilde{q_{ij}}^2 \in \mathbb{G}_3'$.

(22): simply $\chi_R(g_R) = -q_{ii} \neq q_{ii}, q_{jj}, q_{kk}, q_{ll}$ in all the possible cases.

(23): for $t = l$ we have that $\chi_R(g_R) = q_{jj}^3q_{ll} \neq q_{ll}$, and for $t = i, k$ we have $\chi_R(g_j)\chi_j(g_R) = 1 \neq \widetilde{q_{ij}}, \widetilde{q_{kj}}$. Suppose that $\chi_R = \chi_j$ and $g_R = g_j$. Then $\widetilde{q_{ij}} = \chi_R(g_i)\chi_i(g_R) = \widetilde{q_{ij}}^3$, which is a contradiction because $\widetilde{q_{ij}} \neq \pm 1$.

(24): now, $\chi_R(g_i)\chi_i(g_R) = \chi_R(g_j)\chi_j(g_R) = 1 \neq \widetilde{q_{ij}}, \widetilde{q_{kj}}$, so we discard the cases $t = i, j, k$. Now $\widetilde{q_{kl}} = q_{kk}^{-1} \neq \chi_R(g_k)\chi_k(g_R) = q_{kk}$ so also $(\chi_R, g_R) \neq (\chi_l, g_l)$.

(25)**(i)** : again $\chi_R(g_i)\chi_i(g_R) = \chi_R(g_j)\chi_j(g_R) = 1 \neq \widetilde{q_{ij}}, \widetilde{q_{kj}}$, and the cases $t = i, j, k$ are solved. As $\widetilde{q_{kl}} = q_{jj}^3 \neq \chi_R(g_k)\chi_k(g_R) = q_{jj}$, we conclude that $(\chi_R, g_R) \neq (\chi_l, g_l)$.

(26): for $t = j, k$ note that $\chi_R(g_i)\chi_i(g_R) = \widetilde{q_{ij}}\widetilde{q_{ik}} \neq \widetilde{q_{ij}}, \widetilde{q_{ik}}$. For $(\chi_R, g_R) = (\chi_i, g_i)$,

$$q_{ii} = \chi_R(g_R) = -q_{jj}q_{kk}, \quad \widetilde{q_{ij}} = \chi_R(g_j)\chi_j(g_R) = \widetilde{q_{ij}}^3q_{jj}^2, \quad \widetilde{q_{ik}} = \chi_R(g_k)\chi_k(g_R) = \widetilde{q_{ik}}^3q_{kk}^2,$$

so $q_{jj} = -q_{kk} = \pm q_{ii}^2$, but this diagram is not in [19], a contradiction.

(28): we look for the possible generalized Dynkin diagrams for which we need R.

$\circ^{\zeta^4} \overset{\zeta^9}{\text{———}} \circ^{\zeta^8}$, $\zeta \in \mathbb{G}_{12}'$: $\chi_R(g_R) = 1 \neq q_{ii}, q_{jj}$.

$\circ^{\zeta^8} \overset{\zeta}{\text{———}} \circ^{\zeta^8}$, $\zeta \in \mathbb{G}_{12}'$: $\chi_R(g_i)\chi_i(g_R) = \chi_R(g_j)\chi_j(g_R) = \zeta^{10} \neq \widetilde{q_{ij}}$.

$\circ^{-\zeta} \overset{\zeta^7}{\text{———}} \circ^{\zeta^3}$, $\zeta \in \mathbb{G}_9'$: $\chi_R(g_R) = \zeta^8 \neq q_{ii}, q_{jj}$.

$\circ^{\zeta^6} \overset{\zeta^{11}}{\text{———}} \circ^{\zeta^8}$, $\zeta \in \mathbb{G}_{24}'$: $\chi_R(g_R) = \zeta^4 \neq q_{ii}, q_{jj}$.

$\circ^{-\zeta} \overset{-\zeta^{12}}{\text{———}} \circ^{\zeta^5}$, $\zeta \in \mathbb{G}_{15}'$: $\chi_R(g_R) = \zeta^{12} \neq q_{ii}, q_{jj}$.

(29): we consider each possible generalized Dynkin diagram.

$\circ^{-\zeta} \xrightarrow{\ \zeta^3\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_5$: $\chi_R(g_R) = 1 \neq q_{ii}, q_{jj}$.

$\circ^{\zeta^3} \xrightarrow{\ -\zeta^4\ } \circ^{-\zeta^{11}}$, $\zeta \in \mathbb{G}'_{15}$: $\chi_R(g_R) = \zeta^{11} \neq q_{ii}, q_{jj}$.

$\circ^{\zeta^8} \xrightarrow{\ \zeta^3\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_{20}$: $\chi_R(g_R) = \zeta^{12} \neq q_{ii}, q_{jj}$.

$\circ^{\zeta^8} \xrightarrow{\ \zeta^{13}\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_{20}$: $\chi_R(g_R) = \zeta^{12} \neq q_{ii}, q_{jj}$.

$\circ^{-\zeta^3} \xrightarrow{\ \zeta^3\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_7$: $\chi_R(g_R) = \zeta^2 \neq q_{ii}, q_{jj}$.

$\circ^{\zeta^2} \xrightarrow{\ \zeta^3\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_8$: $\chi_R(g_R) = 1 \neq q_{ii}, q_{jj}$.

(30): again consider each possible generalized Dynkin diagram.

$\circ^{\zeta^4} \xrightarrow{\ \zeta^{11}\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_{12}$: $\chi_R(g_R) = \zeta^{10} \neq q_{ii}, q_{jj}$.

$\circ^{\zeta^8} \xrightarrow{\ \zeta^7\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_{12}$: $\chi_R(g_R) = \zeta^2 \neq q_{ii}, q_{jj}$.

$\circ^{\zeta^8} \xrightarrow{\ \zeta^3\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_{24}$: $\chi_R(g_i)\chi_i(g_R) = \zeta$, $\chi_R(g_j)\chi_j(g_R) = 1 \neq \widetilde{q_{ij}}$.

$\circ^{\zeta^6} \xrightarrow{\ \zeta\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_{24}$: $\chi_R(g_R) = \zeta^{15} \neq q_{ii}, q_{jj}$.

$\circ^{-\zeta} \xrightarrow{\ -\zeta^{12}\ } \circ^{\zeta^5}$, $\zeta \in \mathbb{G}'_{15}$: $\chi_R(g_R) = \zeta^{10} \neq q_{ii}, q_{jj}$.

(31): the unique diagram is $\circ^{\zeta^3} \xrightarrow{\ \zeta^8\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_9$, and $\chi_R(g_R) = -\zeta^6 \neq q_{ii}, q_{jj}$.

(32): we consider each possible generalized Dynkin diagram.

$\circ^{\zeta} \xrightarrow{\ \zeta^2\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_5$: $\chi_R(g_R) = 1 \neq q_{ii}, q_{jj}$.

$\circ^{\zeta} \xrightarrow{\ \zeta^{17}\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_{20}$: $\chi_R(g_R) = \zeta^5 \neq q_{ii}, q_{jj}$.

$\circ^{\zeta^{11}} \xrightarrow{\ \zeta^7\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_{20}$: $\chi_R(g_R) = \zeta^{15} \neq q_{ii}, q_{jj}$.

$\circ^{\zeta^3} \xrightarrow{\ -\zeta^4\ } \circ^{-\zeta^{11}}$, $\zeta \in \mathbb{G}'_{15}$: $\chi_R(g_R) = \zeta \neq q_{ii}, q_{jj}$.

$\circ^{\zeta^5} \xrightarrow{\ -\zeta^{13}\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_{15}$: $\chi_R(g_R) = \zeta^{10} \neq q_{ii}, q_{jj}$.

(33): the unique diagram is $\circ^{\zeta^3} \xrightarrow{\ -\zeta^2\ } \circ^{-1}$, $\zeta \in \mathbb{G}'_9$, and $\chi_R(g_R) = \zeta^9 \neq q_{ii}, q_{jj}$. □

Theorem 6.3. *Suppose V is an object in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ such that its Nichols algebra has a finite root system. Then* $\mathrm{Hom}^{\Bbbk\Gamma}_{\Bbbk\Gamma}(\Bbbk\mathcal{R}_V, V) = 0$.

Proof. If $f \in \mathrm{Hom}^{\Bbbk\Gamma}_{\Bbbk\Gamma}(\Bbbk\mathcal{R}_V, V)$ and $R \in \mathcal{R}_V$, then $f(R) \in V^{\chi_R}_{g_R}$. By Proposition 6.2, $V^{\chi_R}_{g_R} = 0$ for each $R \in \mathcal{R}_V$, so $f = 0$. □

Theorem 6.4. *If $\mathcal{B}(V)$ is a Nichols algebra of diagonal type with finite root system then $\mathcal{B}(V)$ does not admit nontrivial graded deformations as a braided bialgebra.*

Proof. We fix a realization of (V, c) in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ where Γ is an abelian group. Without loss of generality, we may assume that the g_i's generate Γ and the χ_i's generate $\widehat{\Gamma}$. By Theorem 6.3

and Remark 4.2, the conditions needed to invoke Theorem 5.3 are satisfied, so $\mathcal{B}(V)$ does not admit nontrivial graded deformations in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$. But our choice of realization ensures that any graded deformation of $\mathcal{B}(V)$ is in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ and hence must be trivial. □

7. Examples

7.1. **Positive parts of quantum groups.** It is well known that, in the generic case, the positive part of a quantized enveloping algebra is a Nichols algebra of diagonal type. By Theorem 6.4, these positive parts are rigid. More generally, this applies to the "diagram" of the pointed Hopf algebra $U(\mathcal{D})$ associated to a generic datum $\mathcal{D}$ of finite Cartan type — see [2], where it is shown that any pointed Hopf algebra whose group-like elements form a finitely generated abelian group is isomorphic to some $U(\mathcal{D})$ if it is a domain with finite Gelfand-Kirillov dimension and its infinitesimal braiding is positive.

7.2. **Distinguished pre-Nichols algebras.** These are infinite-dimensional braided Hopf algebras projecting onto the corresponding finite-dimensional Nichols algebras. They were formally defined in [5, Definition 3.1] generalizing the situation with quantum groups at roots of unity and the corresponding small quantum groups. Let V be a braided vector space of diagonal type such that $\mathcal{B}(V)$ is finite-dimensional. Then the distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}(V)$ is the quotient of $T(V)$ by the relations in Theorem 6.1 except the powers of root vectors (6). As a consequence of Theorem 6.3, we have:

Theorem 7.1. *Let (V, c) be a braided vector space of diagonal type such that $\mathcal{B}(V)$ is finite-dimensional. Then $\widetilde{\mathcal{B}}(V)$ does not admit nontrivial graded deformations as a braided bialgebra.* □

7.3. **Nichols algebras over dihedral groups.** Let D_m denote the dihedral group of order $2m$. For odd m, it is not known whether the category of Yetter-Drinfeld modules over D_m has any finite-dimension Nichols algebras. For even $m \geq 4$, the only known finite-dimensional Nichols algebras have a symmetric braiding [13], so Theorem 3.3 applies.

7.4. **Nichols algebras over symmetric groups.** Let $n \geq 3$. The quadratic algebra $\mathcal{FK}_n$, introduced by Fomin and Kirillov [14], is presented by generators $x_{(ij)}$, $1 \leq i < j \leq n$, and relations

$$\begin{aligned}
x_{(ij)}^2 &= 0, && 1 \leq i < j \leq n,\\
x_{(ij)}x_{(jk)} &= x_{(jk)}x_{(ik)} + x_{(ik)}x_{(ij)}, && 1 \leq i < j < k \leq n,\\
x_{(jk)}x_{(ij)} &= x_{(ik)}x_{(jk)} + x_{(ij)}x_{(ik)}, && 1 \leq i < j < k \leq n,\\
x_{(ij)}x_{(kl)} &= x_{(kl)}x_{(ij)}, && \#\{i, j, k, l\} = 4.
\end{aligned}$$

Milinski and Schneider [26] showed how to make $\mathcal{FK}_n$ a graded bialgebra in the category of Yetter-Drinfeld modules over the symmetric group S_n. As an algebra, it is generated by the vector space V_n with basis $\{x_{(ij)} \mid 1 \leq i < j \leq n\}$. Identifying (ij) with the corresponding transposition in S_n, we can make V_n a Yetter-Drinfeld module where the coaction is defined by declaring x_σ a homogeneous element of degree σ, and the action is the conjugation twisted by the sign. The corresponding braiding on V_n is given by

$$c(x_\sigma \otimes x_\tau) = \chi(\sigma,\tau) x_{\sigma\tau\sigma^{-1}} \otimes x_\sigma, \qquad \chi(\sigma,\tau) = \begin{cases} 1 & \sigma(i) < \sigma(j), \tau = (ij),\, i < j, \\ -1 & \text{otherwise,} \end{cases}$$

where σ and τ are transpositions. Then the above relations generate a biideal in the (braided) tensor bialgebra $T(V_n)$.

It is easy to see that $\mathcal{FK}_n$ projects onto the Nichols algebra $\mathcal{B}(V_n)$. For $n = 3, 4, 5$, it is known that $\mathcal{FK}_n = \mathcal{B}(V_n)$ and has dimension, respectively, 12, 576 and 8294400 (see [26] for $n = 3, 4$ and [17] for $n = 5$). Milinski and Schneider conjectured that $\mathcal{FK}_n$ coincides with $\mathcal{B}(V_n)$ for all n. Moreover, it has been conjectured that $\dim \mathcal{FK}_n = \infty$ for $n \geq 6$ [14].

Theorem 7.2. *Let $n \geq 3$. Then $\mathcal{FK}_n$ does not admit nontrivial graded deformations as a braided bialgebra.*

Proof. All relations are in degree 2 and cannot have coaction given by transposition. As the only primitives in degrees smaller than 2 are in degree 1 and have coaction given by transpositions, the assumption of Theorem 5.3 is satisfied and these algebras are rigid. □

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